Fundamentals of Biomechanics

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Lecture notes at:
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These lectures are intended to give you the basic mechanical theory that you will need to understand the rest of the course. It is necessarily technical but hopefully not too difficult to follow. It will be revision for anyone with A-level maths or physics, and much of it will have been covered in GCSE maths and physics too. You will be assessed on this part of the course by solving a number of problems and there are plenty of examples to work through to give you practice. This part of the course will concentrate on rigid-body mechanics since this is the commonest and most generally useful area.

1) What is biomechanics?

As you can probably guess biomechanics is the application of mechanics to biology. Mechanics is a branch of applied mathematics that deals with movement and tendency to movement; it is also the 'science of machines'. In practice there is no difference between biomechanics and mechanics except what is studied. Certainly in terms of underlying theory there is no difference whatsoever. However common usage of the term varies slightly from this rigid definition. A biomechanist is often interested in the physiology underlying movement (muscle physiology, nervous control, for example) and also the biological rôle of the movement (foraging, ranging, predator avoidance). Additionally certain aspects of mechanics are rarely of interest such as quantum mechanics and relativity.

a) Brief history of biomechanics

Formal mechanics in the modern sense dates back to Sir Isaac Newton in the 17th century but studying objects in motion dates back to the Ancient Greeks. Biology has always had a strong influence on design:

If one way be better than another, that you may be sure is Nature's way.
Aristotle, fourth century B.C.E.

Human ingenuity may make various inventions, but it will never devise any inventions more beautiful, nor more simple, nor more to the purpose than Nature does; because in her inventions nothing is wanting and nothing is superfluous.
Leonardo da Vinci, fifteenth century

Sources of hydraulic contrivances and of mechanical movements are endless in nature; and if machinists would but study in her school, she would lead them to the adoption of the best principles, and the most suitable modifications of them in every possible contingency.
Thomas Ewbank, mid-nineteenth century

One handbook that has not yet gone out of style, and predictably never will, is the handbook of nature. Here, in the totality of biological and bio-chemical systems, the problems mankind faces have already been met and solved, and through analogues, met and solved optimally.
Victor Papanek, contemporary

b) Relevance to ergonomics

A common problem in ergonomics is the analysis of a human performing a given task and the design of appropriate tools. One part of this analysis is to understand the mechanics of the person and any interactions with his or her surroundings – essentially a biomechanical problem. Thus biomechanics is a key skill for the ergonomist.

2) Fundamental concepts

a) Dimensions

Since biomechanics is a quantitative discipline there are a set of units that must be used when expressing values. In fact there are only three basic units that are used and all other units that we encounter can be considered as composites of the basic three. These composite units are often given names to make them less cumbersome to use but remembering how they break down into the basic units can be a useful aid to remembering the underlying equations.

i) Length

Length (or distance) is obviously a key measurement in describing movement. It should always be converted into metres before doing any biomechanical calculations to avoid problems later on. It is generally measured with various forms of calibrated rulers and tapes although it can by more complex methods such as the timing of fixed velocity waves (sound and electromagnetic).

The metre is the length of the path travelled by light in vacuum during a time interval of 1/299 792 458 of a second.

ii) Time

This is another key measurement that allows us to quantify changes of position. Velocity and acceleration are distances differentiated with respect to time. It should always be converted into seconds for calculation. It is generally measured by counting oscillations: springs, pendulums, or electronic oscillators such as crystals.

The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom.

iii) Mass

This allows us to measure how much of a material we are dealing with. Again this is essential for mechanics and allows us to quantify inertia (inertia and mass are basically synonyms). It is usually measured by measuring the force due to gravity that is exerted on the mass. There is currently no way of defining a kilogram except with reference to the world's standard kilogram kept by the International Bureau of Weights and Measures (BIPM) in Paris.

The kilogram is equal to the mass of the international prototype of the kilogram.

b) Newton

Newton published his *Principia* in 1686 which lays down the fundamental rules of mechanics. In this he published his famous three laws of motion that can be used to solve most biomechanical problems. It also contains his law of universal gravity which is also essential.

i) Newton's First Law of Motion

Every body continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it.

What this means is that a stationary body will stay stationary if there are no net forces acting on it: no surprises there. But it also says that a body moving with constant velocity will continue to move at a constant velocity if there are no net forces acting on it which needs a bit more thought. We all know that you have to keep pushing something to keep it moving. However what is happening in this case is that you need to keep applying a force that is equal to and in the opposite direction of the force produced by friction (more on friction later). If you have an object moving at uniform velocity on ice for example it will keep going for a very long time because the friction is very low.

ii) Newton's Second Law of Motion

The change of motion of an object is proportional to the force impressed and is made in the direction of the straight line in which the force is impressed.

This is actually the definition of what a force is: something that causes objects to accelerate. It can be stated mathematically as:

Equation 1.

$$\overline{F} = m\overline{a}$$

Where:

$\overline{F}$ is the net external force (N)
m is the mass of the object (kg)
$\overline{a}$ is the acceleration of the object (ms^{-2})

This means that if you know any two of force, mass and acceleration you can calculate the missing value. Note that force and acceleration are vector quantities: this means that their direction is important not just their magnitude. The second law also explains the first law. If no net force is acting on a mass then there is no change of velocity (acceleration). Thus a stationary object (velocity zero) will stay stationary, and an object with a uniform velocity will continue with that same uniform velocity.

iii) Newton's Third Law of Motion

To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal and directed to contrary parts.

What this means is that whenever a force is applied to an object then force acts both ways. If I push a wall with a force of 100N then the wall pushes back at me with exactly the same force. Again this is often counter intuitive because often only one of the objects appears to move but once again this is generally because of friction. If you stand in a boat and push against a wall you will accelerate backwards. If you push another boat, both boats will move away from each other. The earth does move slightly when you push against it but because it is so much more massive than you are you do not see the movement!

iv) Newton's Law of Universal Gravitation

This is the law that is supposed to relate to the apple falling incident. Newton stated that every object attracts every other object with a force inversely proportional to the square of the distance between the two objects and proportional to the mass of each of the objects. Mathematically this can be stated as:

Equation 2.

$$\overline{F} = G\left(\frac{m_1 m_2}{r^2}\right)$$

Where:

$\overline{F}$ is the gravitational force acting between the objects (N)
G is universal constant of gravitation ($m^3s^{-2}kg^{-1}$)
m_1 is the mass of object 1 (kg)
m_2 is the mass of object 2 (kg)
r is the distance between the objects (m)

G is very small (6.67259×10^{-11} $m^3s^{-2}kg^{-1}$) so in practice unless one of the objects has a very high mass we can ignore this force. The only important large mass for biomechanics is the mass of the earth. In this case the distance is also fixed since the radius of the earth is very much larger than any normal changes of altitude. This means that we can create a new constant for use on the surface of the earth which we call *g*.

Equation 3.

$$\overline{g} = G\left(\frac{m_2}{r^2}\right)$$

Where:

$\overline{g}$	is the acceleration due to gravity (m s^{-2})
G	is universal constant of gravitation ($m^3 s^{-2} kg^{-1}$)
m_2	is the mass of the earth (kg)
r	is the radius of the earth (m)

And we can substitute this back into equation 2 to get:

Equation 4.

$$\overline{F} = m_1 \overline{g}$$

Which is just a restating of equation 1. This means that the force due to gravity acting on an object on the surface of the earth is simply its mass multiplied by g. g varies slightly in different parts of the world but the standard UK value is $9.81 ms^{-2}$. This force is always present and always acts directly downwards (it defines the direction of down). A matching force acts upwards on the earth itself and an object will tend to accelerate towards the centre of the earth if no other forces act on it. This acceleration generally ceases when the object hits the surface of the earth since the contact generates a ground reaction force equal and opposite to the gravitational force.

3) Forces

Forces are key to understanding mechanics. The unit of force is the Newton which is equivalent to $1 kgms^{-2}$. You will sometimes see forces measured as kg weight (or even lb weight) which is the force needed to lift that weight in kg. It is poor practice to use kg weight in a scientific context since it depends on the local value of g which does vary by a few percentage points around the globe. However it is often easier for non-scientists to understand: people have a feel for how much force it takes to lift a 1kg bag of sugar but do not know how big 10N is. You convert from kg weight to Newtons by multiplying by g.

a) Internal

When we investigate a biomechanical problem we are usually considering a body acting within an environment. The forces we are considering can be internal to the body or external. Internal forces are the forces that act within the body: muscle forces, joint reaction forces, loads that act on the various body tissues. These forces cause the body shape to change by moving the various segments (limbs, torso, head) relative to each other. However in themselves they do not move the body. If you were in free fall you could move your body however you liked and all you would do would be to spin in the same place (or continue to move with a uniform velocity if you started off with a non-zero velocity).

b) External

To move relative to the outside world the body needs to be subject to external forces. These are often the result of internal forces causing a change in the body conformation but can also be due to other external forces such as gravity or externally applied forces from contact with other objects.

Contact forces can be divided into two components. The first component acts perpendicularly to the contact surface and is called the normal contact force (normal in this context simply means perpendicular). The other force acts tangentially to the contact surface and is called friction.

c) Friction

Sadly friction is a very complex phenomenon. There are certain simplifications that sometimes work approximately and are a good starting point. One thing that is always true is that the friction force always opposes the direction of the applied force: it always slows things down and makes them harder to get moving in the first place. If an object is resting on a smooth surface and pushed from the side it will take a certain amount of force to get it moving. The force required to keep it moving is usually then rather less. If extra weight is added on top of the object the force required to get the object moving and to keep it moving will go up. Oddly enough, for many materials, changing the surface area of contact does NOT change the force required to make the object slip. This is rather counter-intuitive since we think of wide car tyres having a better grip on the road – this is because rubber is one of the materials where friction is area dependent. The force required to overcome friction varies more or less linearly with the normal contact force. The force required to get an object moving from stationary is the static friction force. The lower force required to keep a moving object moving is the dynamic friction force.

We can express these relationships mathematically:

Equation 5.

$$F_s = \mu_s N$$

Where:

F_s	is the magnitude of the static friction force (N)
μ_s	is the coefficient of static friction
N	is the magnitude of the normal contact force (N)

Equation 6.

$$F_d = \mu_d N$$

Where:

F_d	is the magnitude of the dynamic friction force (N)
μ_d	is the coefficient of dynamic friction
N	is the magnitude of the normal contact force (N)

d) Adding Forces

Forces are vector quantities: they have magnitude and direction. Almost always there will be more than one force acting on a body and very commonly we want to find the single equivalent force. Forces can simply be summed as vectors and the summed value is known as the resultant (or net) force.

i) Colinear

Quite commonly we can arrange the problem so that the forces act in a straight line. If this is the case then we call the forces collinear and we can sum them numerically. The sign represents the direction of the force and we simply define positive as one direction along the line and negative as the other direction.

ii) Concurrent

In other situations the forces do not act in a single line but all act on a single point. In this case we can use vector arithmetic to sum the forces. If the forces act at right angles we can simply treat the right-angled components as separate collinear forces. In the general case we can use trigonometry to convert the force vectors into x and y components, sum these components separately, and then use trigonometry to convert back to a magnitude and direction. If we are not too concerned about accuracy we can do this calculation by drawing the forces as arrows pointing in the correct directions whose length is scaled to the magnitude of the force. These arrows are joined end to end and the resultant force is the arrow that is required to move from the start to the end point in a single step.

iii) General Case

In the general case, forces acting in miscellaneous directions and not all through a single point, there will be a rotational component. The non-rotational (linear) component can be calculated exactly as indicated above but details of how to calculate the rotational component will be covered later.

e) Static Equilibrium

It follows on from Newton's laws that if a body is not moving (or moving at constant velocity) then the sum of all forces in the system is zero. In practical terms if a body is only accelerating slowly and an approximate answer is sufficient (as is often the case in ergonomics) then the problem can often be considered one of static equilibrium even if this is not strictly the case. This sort of approximation tends to underestimate the forces and can be used to estimate a lower bound. We often use static equilibrium to estimate internal forces such as the forces in muscle and joints since these are difficult to measure directly.

f) Free Body Diagram

The first step is to draw a free body diagram. This is a picture of the problem with all the relevant forces (both known and unknown) drawn in. A good diagram is an essential first step for handling almost any biomechanical problem since even if it is not directly used for analysis it will clarify the problem and make it much less likely that a key component will be ignored. The free body diagram will always include the gravitational force acting at the centre of mass. It will also include any external forces that are acting (although small forces such air resistance and air buoyancy are usually ignored) and any internal forces that are relevant (what is and is not relevant will depend on the question that is being answered). Known forces can be written in as their values in Newtons (and their direction if relevant), unknown forces should be represented as letters.

g) Static Analysis

In static analysis the total force will add up to zero. This means that the known forces and the unknown forces can be added together and the answer will equal 0. If there is only one unknown force this can equation can be rearranged to solve the unknown. If there are two unknown values we may be able to divide the forces into two sets acting perpendicularly. Remember that perpendicular forces act independently so both these sets must add up to zero and this may allow us to solve the problem. If the forces are not concurrent we may also be able to use the rotational components as described later.

4) Linear Kinematics

Kinematics is the subsection of mechanics that describe how an object moves: position, velocity and acceleration. Linear kinematics describe objects that move in straight lines. Usually this is a simplification of an actual problem but many problems can be considered as movement in a straight line.

a) Rectilinear

Straight-line motion is also called rectilinear motion. All points of the body move in a straight line and there is no orientation change of the individual components (no rotation).

b) Curvilinear

This is an extension of rectilinear motion. The object's orientation still does not change but the direction of motion does change. If you think about it this is a rather unlikely action for a human since we normally face the direction that we are moving so that we rotate as we change direction but it might apply in jumping forward: the subject always faces forward but the trajectory starts off going upwards and forwards and gradually changes to going downwards and forwards.

c) Angular

In pure angular motion an object rotates about its centre of mass: a dancer pirouetting for example has angular motion with no linear displacement.

d) Composite

In general movement consists of a composite of both angular and linear movement. Fortunately these components can be isolated and treated independently to a great extent.

e) Cartesian Coordinates

The commonest way of representing linear kinematics is to use Cartesian (x, y) coordinates to represent position relative to a fixed origin. This has the advantage that the x and y values can often be treated independently since they are perpendicular (and it is easy to draw any values on a graph too). Generally x is used to represent forward movement and y vertical movement. If you are lucky only one axis is used in any particular problem.

f) Displacement

Displacement is used to represent the position. This is the distance in metres in the x and y direction from the origin.

g) Speed

If the position is changing over time (i.e. an object is moving) then we can calculate the average speed as the distance moved divided by the time taken. We can also calculate the instantaneous speed by calculating the gradient of the distance time graph (in practice that can be measured with a speedometer or calculated by calculating the distance moved in a very short time interval). Speed is often (and wrongly) used interchangeably with velocity. Speed is a scalar value that does not take into account the direction whereas the velocity is a vector quantity with both speed and direction. When we are dealing with rectilinear problems since the direction does not change the two values are almost the same except that you can have positive and negative velocities but speed is always positive. Speed is measured in ms^{-1}. In general you should avoid using the term speed – you almost always want to use velocity since the direction is always important in mechanics!

h) Velocity

Velocity is the vector quantity representing both speed and direction. The distinction between speed and velocity is vitally important in curvilinear problems. It is measured in ms^{-1} but should also have a direction component (x, y, or an angle).

i) Acceleration

If velocity is changing over time then we can calculate the average acceleration as the change of velocity divided by the time taken (0 to 60 mph in 5 seconds is $5.4ms^{-2}$). It is measured in ms^{-2} and again should have a direction. Occasionally acceleration is measured in multiples of g (the acceleration due to gravity). The same caveats apply as with kg weight and so-called g-force is used for the same reason. People are more familiar with the acceleration they experience when falling (1g) than with $10ms^{-2}$ but its value is imprecise because g varies. To convert from g divide by 9.81 ms^{-2}. Instantaneous acceleration can be measured with an accelerometer or can be calculated by measuring the change in velocity in a very small time interval.

There are some important relationships between displacement, velocity and acceleration for constant accelerations acting in a straight line. We define the key values as follows:

a	is the magnitude of the constant acceleration (ms^{-2})
u	is the magnitude of the start velocity (ms^{-1})
v	is the magnitude of the end velocity (ms^{-1})
t	is the time (s)
s	is the magnitude of the displacement (m)

The acceleration is the gradient of the graph of velocity against time which can be calculated as:

Equation 7.

$$a = \frac{v - u}{t}$$

The displacement is the area underneath the graph of velocity against time which can be calculated as:

Equation 8.

$$s = \frac{(u + v)t}{2}$$

You can use these two equations to eliminate one of the variables. Thus for example:

Eliminating v

Equation 9.

$$s = ut + \frac{1}{2}at^2$$

Eliminating t

Equation 10.

$$v^2 = u^2 + 2as$$

5) Linear Kinetics

Kinematics allows us to describe how an object is moving. If we then add the forces that are causing it to move we use the subsection of mechanics called kinetics. This is really where we start to use Newton's Laws of Motion since they involve forces.

a) Zero Net Force

In Newton's first law we saw that when there was no net force the velocity of an object remained unchanged. Thus the forces on a stationary object must sum to zero as we have already seen. If we consider the case of projectiles: objects or people thrown or falling we can see that the only force acting on the projectile is gravity acting downwards. Remembering that perpendicular forces can be considered independent that means that the projectile will accelerate downwards but since there is no force acting horizontally (except for a usually negligible amount of air resistance) the horizontal velocity will be constant.

b) Law of Acceleration

Newton's second law allows us to discover the acceleration if we know the resultant force acting on a mass, or alternatively the resultant force of we know the acceleration (or even lets us calculate the mass if we know the force and acceleration). Thus when we accelerate rapidly in a car we can feel the increased force pushing us forwards. If we know the acceleration which we can easily work out from the rate of change of speed we can calculate how much force is acting on us.

c) Impulse and Momentum

There is an alternative mathematical expression representing Newton's second law. The area under a graph of force plotted against time is known as the impulse and this impulse equates to the change in an object's momentum. Momentum is a vector quantity which is the product of the mass and the velocity. If an average value for the force is used (or the actual value if the force is constant) we can use the equation:

Equation 11.

$$\overline{F}t = m\overline{v} - m\overline{u}$$

Where:

$\overline{F}$	is the force (N)
t	is the time (s)
m	is the mass (kg)
$\overline{v}$	is the final velocity (ms^{-1})
$\overline{u}$	is the initial velocity (ms^{-1})

This is a very useful equation because it allows us to calculate the average force that was applied in a time period when we know the change in velocity.

A corollary of this equation is that when no external force is applied to a system the momentum of the system does not change. This is known as *conservation of momentum*.

d) Action and Reaction

Newton's third law means that whenever I push anything, it pushes back at me with exactly the same force but in the opposite direction. If we think about the example of the accelerating car again, the car pushes on me to accelerate me forward with the rest of the contents of the car. Because of Newton's third law I press back an equal amount on the car seat and bend the seat springs accordingly. The car types push backwards on the road and the road pushes forward on the car tyres: the car is accelerated forwards and the earth is accelerated backwards (an insignificant amount). Forces always occur in pairs. You cannot apply a force to a system without the system applying a force back out. However when these other forces act on the earth (such as the force of gravity and reaction forces on the ground) we can ignore their effect on the earth. We also often talk about applying an external force and in this case we simply do not care what happens to the object that is applying the force – generally the object is either firmly attached to the ground or sufficiently massive that its movements are negligible.

6) Work, Power and Energy

Newton's equations are not the only way of calculating forces and movements. We can also use the principle of conservation of energy. In some situations we know the energy transformation that happens during an event and can use this to calculate the outcome.

a) Work

If I transfer energy from myself to an object I do work on that object. The work I do is the area underneath the graph of force against displacement (assuming that the force and displacement are in the same direction). If the force is constant and acting in the direction of the displacement we can use the equation:

Equation 12.

$$W = Fs$$

W is the work done (J)
F is the magnitude of the Force (N)
s is the magnitude of the displacement (m)

Work is measured in Joules (or Newton metres). You will occasionally see work measured in other units such as calories (one calorie is 4.2J) but you should always use Joules. Note that work is a scalar quatity.

b) Energy

Energy is defined as the capacity to do work. If work is done on an object it gains energy. If an object does work it loses energy. Energy is measured in Joules just like work and doing work transforms energy from one form to another. There are lots of forms of energy but the following four are important in biomechanics.

i) Kinetic

Kinetic energy is the energy possessed by objects in motion. A moving object is able to do work and in doing so it will slow down and you need to do work on an object to speed it up. It turns out the amount of energy a moving object has depends on its mass and the square of its velocity. The actual equation is:

Equation 13.

$$E_{KE} = \frac{1}{2}mv^2$$

Where:

E_{KE} is the kinetic energy (J)
m is the mass (kg)
v is the magnitude of the velocity (ms^{-1})

ii) Potential

Potential energy is energy that an object possesses due to its position or shape. There are two types commonly encountered in biomechanics. *Gravitational potential energy* depends on the position of an object in the earth's gravitational field: objects that are high up can do work by allowing themselves to move downwards. Work has to be done on an object to raise it higher. The amount of work depends on the mass and the height change. The equation is:

Equation 14.

$$E_{PE} = mgh$$

Where:

E_{PE}	is the gravitational potential energy (J)
m	is the mass (kg)
g	is the acceleration due to gravity (ms^{-2})
h	is the height (m)

Strain energy is energy stored by elastic deformation of an object. It takes work to stretch a spring and work will be performed if a spring is allowed to relax. The work depends on the force required to perform the shape change and this depends on the properties of the material. Lots of materials have approximately linear spring constants (the graph of force against stretch is a straight line) and in which case the strain energy is given by the following equation:

Equation 15.

$$E_{SE} = \frac{1}{2}k\Delta x^2$$

Where:

E_{SE}	is the strain energy (J)
k	is the spring constant of the material (Nm^{-1})
Δx	is the length change (m)

c) Conservation of Energy

The utility of these relationships is that during an activity energy is conserved and we ignore energy that is transformed into heat due to friction we can use these relationships to calculate mechanical parameters. For example if I fall from a given height I am performing work using gravitational potential energy. This energy is converted into kinetic energy whilst I am falling and then converted into elastic energy when I hit the ground. This means I can calculate the speed I reach for falling a given height and the deformation of the substrate (if I know its spring constant). Sometimes this is simpler than using Newton's equations of motion.

d) Power

Another parameter I might want to know is the power. This defined as the rate of doing work and is measured in Watts or Joules per second. It is a scalar quantity. The equation for average power is:

Equation 16.

$$P = \frac{W}{t}$$

Where:

P	is the power (W)
W	is the work (J)
t	is the time (s)

Power can also be measured instantaneously as the product of force and velocity if they are both acting in the same direction.

Equation 17.

$$P = Fv$$

P	is the power (J)
F	is the magnitude of the Force (N)
v	is the magnitude of the velocity (ms^{-1})

7) Torques and Moments

Much of the movement in the human body is rotational. Limb segments rotate at joints and muscles apply torques and the skeleton acts as a system of levers. This means that we need to know about rotational movement. As you will see rotational movement is very similar to linear movements with rotational analogues for the quantities we measured for linear motion. A complete description of movement of a body needs to include both linear and rotational components and these can be largely treated separately.

a) Torques

Torques are the rotational equivalent of forces. The unit of torque is the Newton□metre and it is defined as a force acting at a distance from an axis of rotation. The distance is the perpendicular distance from the line of action of the force to the axis of rotation (this is also the shortest distance between the line of action of the force and the axis of rotation).

Equation 18.

$$T = Fr$$

T is the torque (Nm)
F is the magnitude of the Force (N)
r is the perpendicular distance from the axis of rotation (m)

In the body muscles apply linear forces at their attachment points. The shortest distance of the line of action of the force from the joint axis is also known as the moment arm of the muscle. The torque applied by the muscle is the product of its tension and the moment arm. It is often the case that the moment arm of a muscle changes as the degree of flexion or extension at a joint changes. Thus the maximum torque that can be generated also changes.

b) Adding Torques

Torques that act around the same axis of rotation can simply be added or subtracted depending on whether they act clockwise or anti-clockwise. Anti-clockwise torques are generally considered positive but this is not always the case and it does not matter as long as you are consistent. If an object is in static equilibrium then the torques sum to zero in much the same way as the linear forces sum to zero.

c) Estimating Muscle Force

In many situations we can measure the external load acting on a body. Since we know where the load is compared to the axis of a given joint we can calculate the torque that the load is applying. If the body is in static equilibrium then we know that the sum of the torques around a joint must be zero. We can therefore estimate the torque produced around a joint by musclular activity. We can estimate the moment arm of the muscles around the joint and therefore can calculate the tension that must be generated by the muscle. This very simple approach ignores the forces due to gravity acting on the body segments but enables us to quickly make a rough estimate of muscle forces.

d) Centre of Mass

Newton's laws all assume that the mass of an object is at a single point in space. This is obviously not true but it turns out that any object or system of linked objects can be considered as a being concentrated at a single point in space as far as linear motion goes. This point is the centre of mass of an object. In regular shaped objects its location is usually obvious (it is at the centre of a sphere for example) but its location is more difficult to work out in irregular objects such as human limbs. It can be obtained experimentally by hanging the segment from a number of different points. The centre of mass is always directly vertically below the suspension point so a line can be draw vertically from the suspension point. This is repeated for a number of suspension points and the centre of gravity is where the lines cross. It can also be calculated mathematically by dividing the shape into a number of smaller shapes. These small shapes are regularly shaped (usually small cubes) so their mass and centre of mass is easy to calculate. The centre of mass of a composite shape is defined by the following equation:

Equation 19.

$$\overline{P}_{CM} = \frac{\sum m_i \overline{P}_i}{\sum m_i}$$

Where:

$\overline{P}_{CM}$	is the position vector of the overall centre of mass (m)
m_i	is the mass of the ith subunit (kg)
$\overline{P}_i$	is the position vector of the ith subunit (m)

The position vectors are simple the X and Y coordinates and these can be treated independently. This approach is very useful for calculating the centre of mass from CT scan data.

e) Centre of Gravity of Human Body

We often wish to know the centre of mass of the whole of the human body. We can measure this using the suspension technique mentioned above but it is often easier to calculate it. The positions of the centres of mass and the mass proportions are known for the major body segments. Thus we can treat these as subunits in equation 19 and can calculate the centre of mass for the whole human body. This technique allows us to calculate the position of the centre of mass whatever the position of the limbs which is extremely useful since subjects are rarely standing in the anatomical position.

8) Angular Kinematics

Now we are interested in rotational movement we need to be able to describe orientation as well as position. By choosing the right units we can create angular analogues of the equations we have previously covered.

a) Angular Position

Angular position is simply the angle that an object is in relation to another object. If that other object is considered immovable (such as the earth) then this is an absolute angle. If both objects are moveable then the angle is relative. Traditionally angles are measured in degrees (°): there are 360 degrees in a circle. However it turns out that calculations are much simpler if we convert our angular measurements to radians and there are 2π radians in a circle. To convert from degrees to radians you divide by $180/\pi$ (and to convert the other way around you multiply by $180/\pi$). In mechanics angles are generally measure anticlockwise from the horizontal. This is different from map bearings which are measured clockwise from North (which is usually vertical).

b) Angular Velocity

If something is rotating its angular position changes with time. This is the gradient of a plot of angle against time and is measured in radians per second (rad·s^{-1}). Average angular velocity can be obtained by dividing the change in angle by the time interval. Rotational speed is sometimes expressed in revolutions per second and this can be converted to radians per second by multiplying by 2π.

Often we want to know the instantaneous tangential velocity of a point on a rotating object. It turns out that this is simply equal to the angular velocity in rad·s^{-1} multiplied by the distance of the point from the axis of rotation.

Equation 20.

$$v = \omega r$$

Where:

v is the velocity tangentially to the circular path of the point (ms^{-1})
ω is the angular velocity (rad s^{-1})
r is the radius of the circular path of the point (m)

c) Angular Acceleration

Angular acceleration is slightly more complex than you might first imagine. Angular acceleration itself is simply the rate of change of angular velocity. It is calculated as the gradient of the angular velocity against time graph, or the change in angular velocity in a time interval. However it becomes more complicated when we want to calculate the instantaneous linear acceleration of a point moving in a circle. It turns out that this can be divided into two separate parts.

i) Tangential Acceleration

Tangential acceleration is simply and extension of the tangential velocity equation and is the angular acceleration multiplied by the distance from the axis.

Equation 21.

$$a_T = \alpha r$$

Where:

a_T is the acceleration tangentially to the circular path of the point (ms^{-2})

α is the angular acceleration (rad s^{-2})
r is the radius of the circular path of the point (m)

ii) Centripetal Acceleration

There is something else that happens to objects moving in circles. This is an acceleration that depends on the angular velocity they are moving rather than the acceleration. You will remember that velocity always includes a direction component and the direction that an object is moving in when it goes round in a circle is constantly changing even though its speed may remain constant. This is a real acceleration (and from Newton's second law must be associated with a real force). It is known as the centripetal acceleration and it is directed towards the centre of the circular path. Its value can be calculated by the following equation:

Equation 22.

$$a_r = \omega^2 r$$

Where:

a_r is the acceleration radially towards the centre of the circular path of the point (ms^{-2})
ω is the angular velocity (rad s^{-1})
r is the radius of the circular path of the point (m)

9) Angular Kinetics

Now we have the tools to describe rotation we can investigate the relationships between torques and rotational movements. As we might expect these are directly analogous to the linear relationships.

a) Moment of Inertia

It turns out that we cannot simply use mass to represent how easy an object is to rotate. Instead we need to use a quantity called the moment of inertia. This quantity is the rotary equivalent of inertia and depends not only on the mass of an object but also on its shape. For a point mass it is simply the mass multiplied by the square of its distance away from the axis of rotation but sadly we cannot just use the centre of mass because if you remember from equation 19 it is dependent on position not position squared.

The equation for moment of inertia is:

Equation 23:

$$I_a = \sum m_i r_i^2$$

Where:

I_a is the moment of inertia about axis a (kgm^2)
m_i is the mass of the i^{th} subunit (kg)
r is the distance of the i^{th} subunit from the axis (m)

It can also be calculated experimentally by allowing the object to act as a pendulum and allowing it to pivot about the axis. Standard values for human body segments are found in anthropometry data books. Since the value depends on the distance from the axis of rotation you need to check that you are using the correct value for the axis you are interested in.

b) Angular Interpretation of Newton's Laws

Now we can use these tools to provide rotational equivalents for Newton's laws of motion.

i) Newton's First Law of Motion

The angular momentum of an object remains constant unless a net external torque is exerted on it.

Angular momentum is the product of angular velocity and moment of inertia. It needs to be used instead of velocity in the linear statement of Newton's first law because whilst mass is unlikely to change and hence is not needed in the law, moment of inertia is easy to change. Since it depends on mass and shape, all the object (for example a human body) needs to do to alter its moment of inertia is change its shape. What this means is that an object that reduces its moment of inertia will increase its angular velocity (and *visa versa*) to maintain its angular momentum unless a next external torque is applied. This is how skaters are able to increase their spin speed by moving their arms closer into their bodies.

ii) Newton's Second Law of Motion

The change in angular momentum of an object is proportional to the net external torque exerted on it, and this change is in the direction of the net external torque. The net external torque is proportional to the rate of change of angular momentum.

Mathematically this is written as:

Equation 24:

$$T_a = I_a\alpha$$

Where:

T_a is the torque about axis a (Nm)
I_a is the moment of inertia about axis a (kgm^2)
α is the angular acceleration about axis a (rad s^{-2})

iii) Newton's Third Law of Motion

For every torque exerted by one body on another, the other body exerts an equal torque back on the first body but in the opposite direction.

This is exactly equivalent to the linear law. The axis of the torque is the same on both bodies.

c) Angular Impulse and Angular Momentum

Just like linear momentum, the angular momentum of a system is conserved. Changes in angular momentum only occur if there is an external angular inpulse (following on from the second law as before).

Equation 25:

$$T_a t = I_a \omega_2 - I_a \omega_1$$

Where:

T_a	is the torque about axis a (Nm)
t	is the time (s)
I_a	is the moment of inertia about axis a (kgm^2)
ω_1	is the initial angular velocity about axis a (rad s^{-1})
ω_2	is the final angular velocity about axis a (rad s^{-1})

10) Not Covered!

This is not an exhaustive coverage of biomechanics by any means. There is no attempt at deriving the equations presented nor a full exploration of their ramifications. However it should provide a good starting point and cover what you need to know for the rest of the lecture series. There are some specific omissions which you may wish to pursue in further reading:

a) 3D

The equations here are generally 2D simplifications. People and their environments are 3D. Generally speaking most ergonomics problems can be simplified to 2D without too much loss of accuracy however there are occasions where 3D is needed. The linear equations are generally vector based so this is just a matter of using 3D rather than 2D vectors but vector algebra will be necessary. The story for rotational movements is much more complex since moment of inertia becomes a 3 by 3 matrix! However the descriptive versions of Newton's laws are quite general.

b) Basic Fluid Mechanics

We generally ignore air resistance but at high speeds this can lead to large errors. If the problem involves moving underwater then fluid effects (buoyancy, friction) are obviously important. This topic is complex although there are some simplifications that can be used for approximate answers.

c) Basic Mechanics of Biological Materials

Material properties (how things bend, stretch and break) are obviously important. Biological materials tend to act in very non-linear fashions which makes this rather more complex than it is for standard engineering materials.