Section 16: Neutral Axis and Parallel Axis Theorem

Geometry of deformation

- We will consider the deformation of an ideal, isotropic *prismatic* beam
  - the cross section is symmetric about y-axis
- All parts of the beam that were originally aligned with the longitudinal axis bend into circular arcs
  - plane sections of the beam remain plane and perpendicular to the beam's curved axis

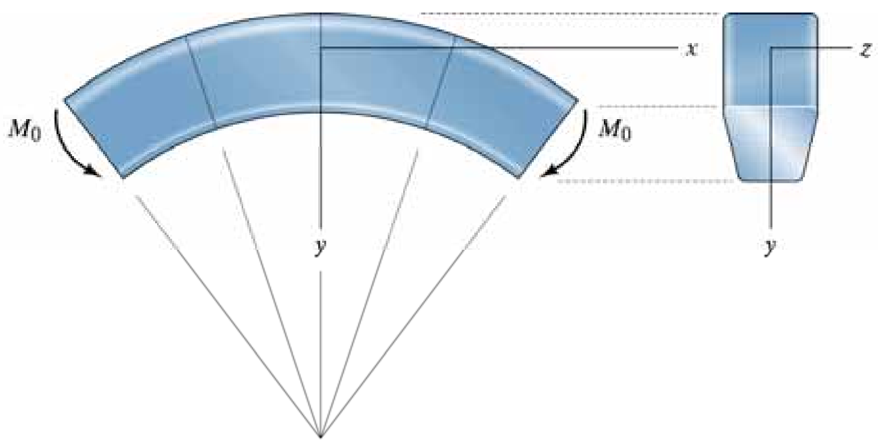


FIGURE 8-3 Deformation resulting from the applied couples. Each cross section of the beam remains plane.

Bedford/Liechti, Mechanics of Materials, 1e, ©2001, Prentice Hall

Note: we will take these directions for M_0 to be positive. However, they are in the opposite direction to our convention (Beam 7), and we must remember to account for this at the end.

Neutral axis

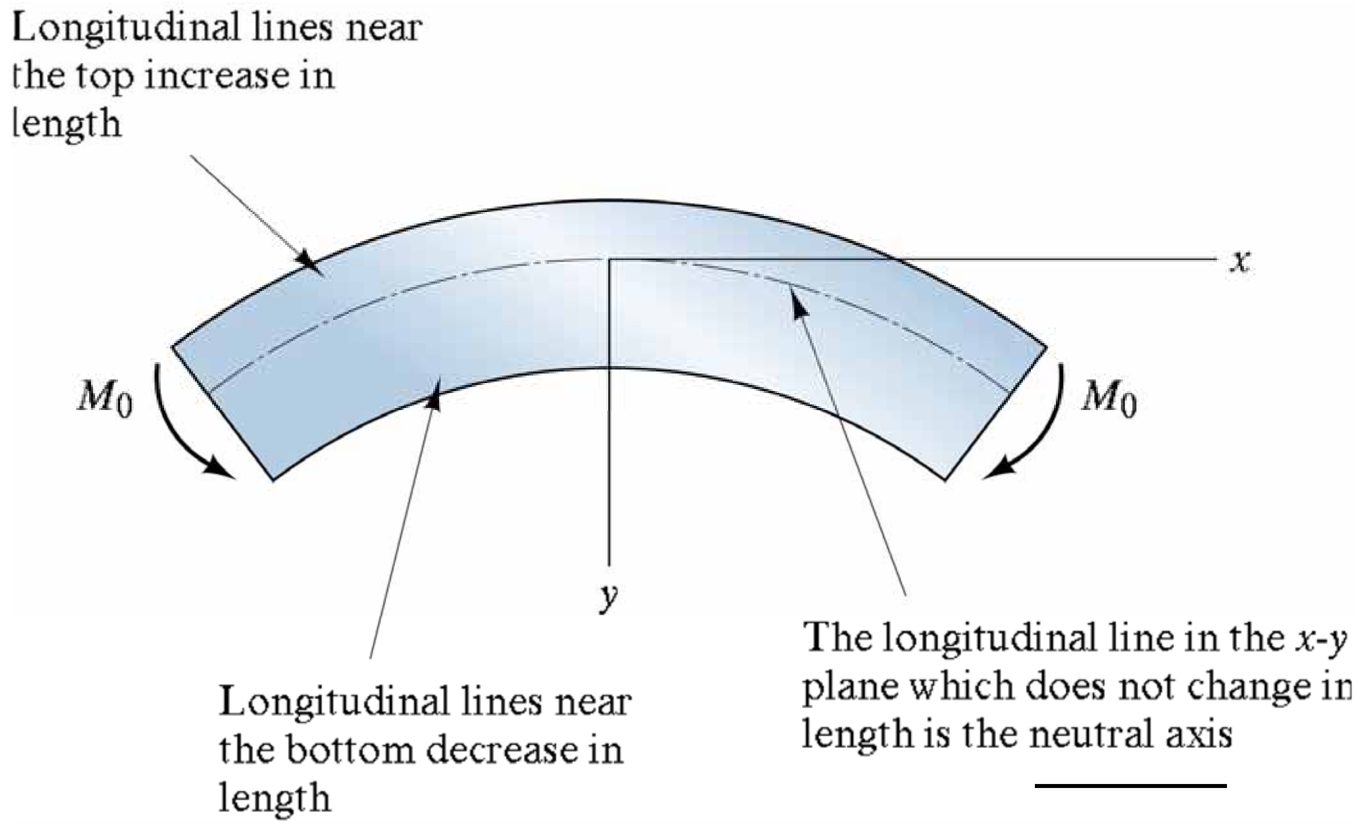
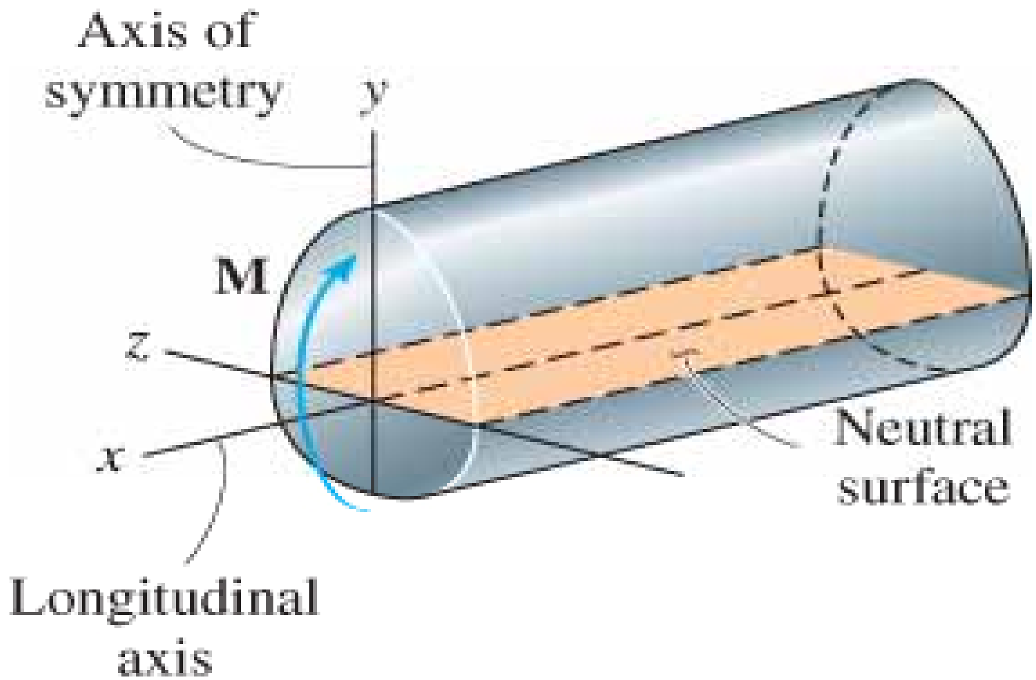


FIGURE 8-4 Changes in the lengths of longitudinal lines.

From: Hornsey

- A *neutral surface* is where longitudinal fibers of the material will not undergo a change in length.

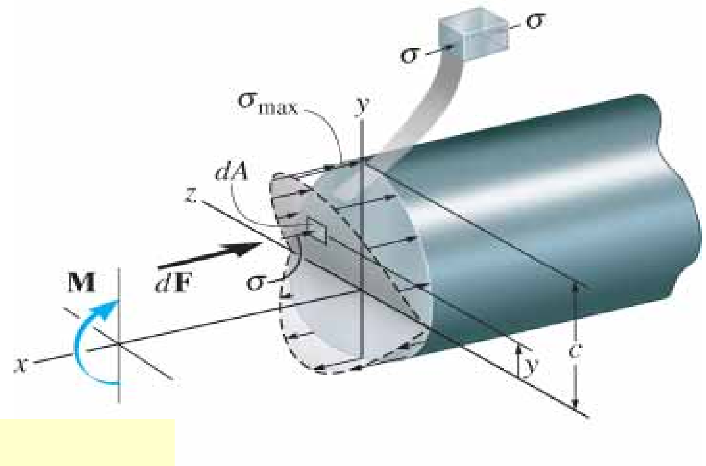


From: Wang

- Thus, we make the following assumptions:
  1. Longitudinal axis *x* (within neutral surface) does not experience any *change in length*
  2. All *cross sections* of the beam *remain plane* and perpendicular to longitudinal axis during the deformation
  3. Any *deformation* of the *cross-section* within its own plane will be *neglected*
- In particular, the *z* axis, in plane of x-section and about which the x-section rotates, is called the *neutral axis*

- By mathematical expression, equilibrium equations of moment and forces, we get

Equation 6-10 $$\int_A y \, dA = 0$$



Bending stress variation

(c)

Equation 6-11 $$M = \frac{\sigma_{max}}{c} \int_A y^2 \, dA$$

- The integral represents the *moment of inertia* of x-sectional area, computed about the neutral axis. We symbolize its value as I.

From: Wang

- Normal stress at intermediate distance y can be determined from

Equation 6-13 $$\sigma = -\frac{My}{I}$$

- σ is -ve as it acts in the -ve direction (compression)
- Equations 6-12 and 6-13 are often referred to as the *flexure formula*.

- Beams constructed of two or more different materials are called composite beams
- Engineers design beams in this manner to develop a more efficient means for carrying applied loads
- Flexure formula cannot be applied directly to determine normal stress in a composite beam
- Thus a method will be developed to “transform” a beam’s x-section into one made of a single material, then we can apply the flexure formula

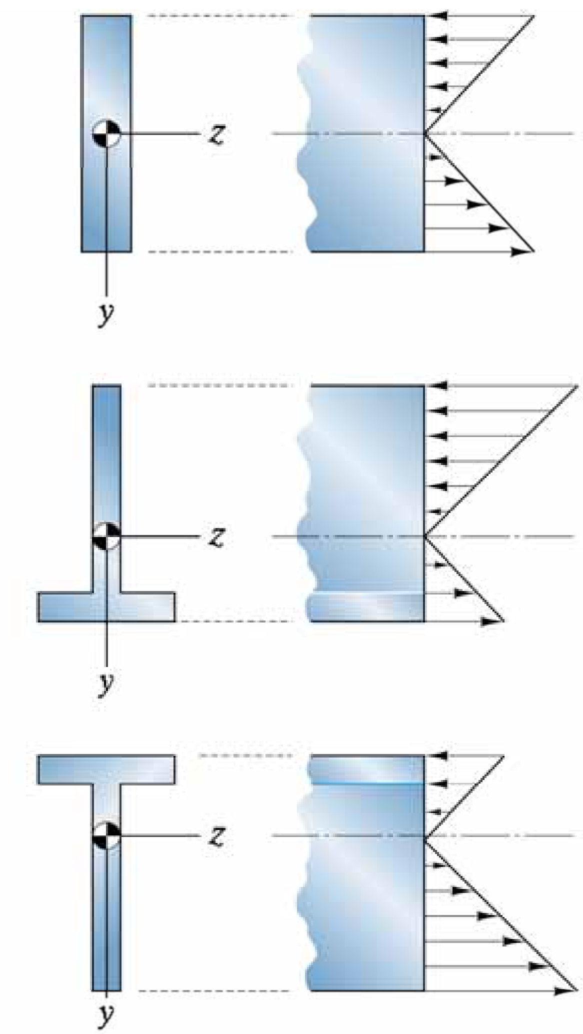


FIGURE 8-15 A T cross section can be used to decrease either the maximum tensile stress or the maximum compressive stress to which a beam is subjected.

Bedford/Liechti, Mechanics of Materials, 1e, ©2001, Prentice Hall

From: Hornsey

Moments of Inertia

- Resistance to bending, twisting, compression or tension of an object is a function of its shape
- Relationship of applied force to distribution of mass (shape) with respect to an axis.

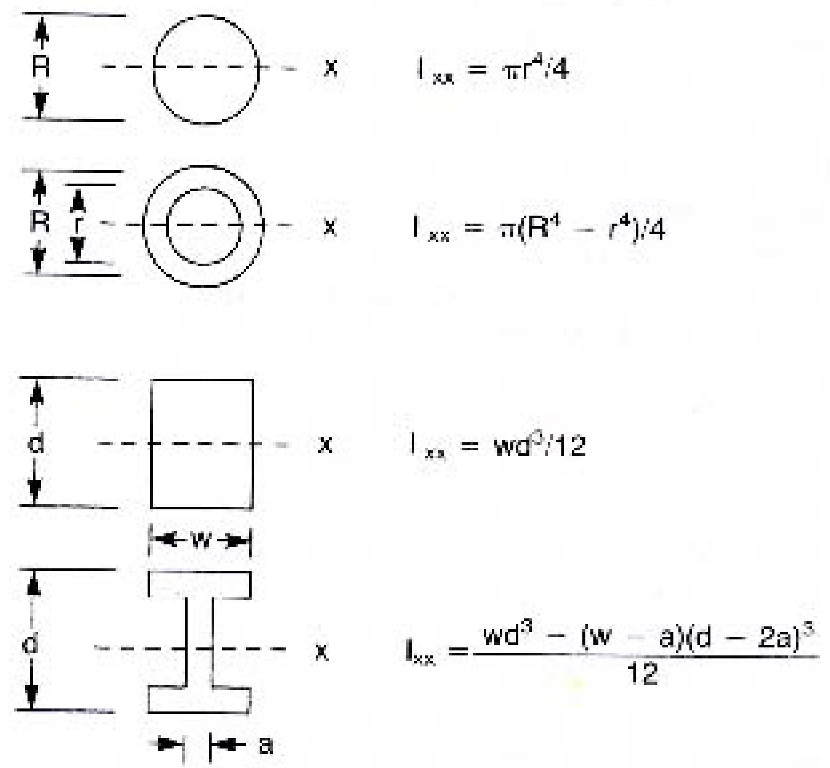
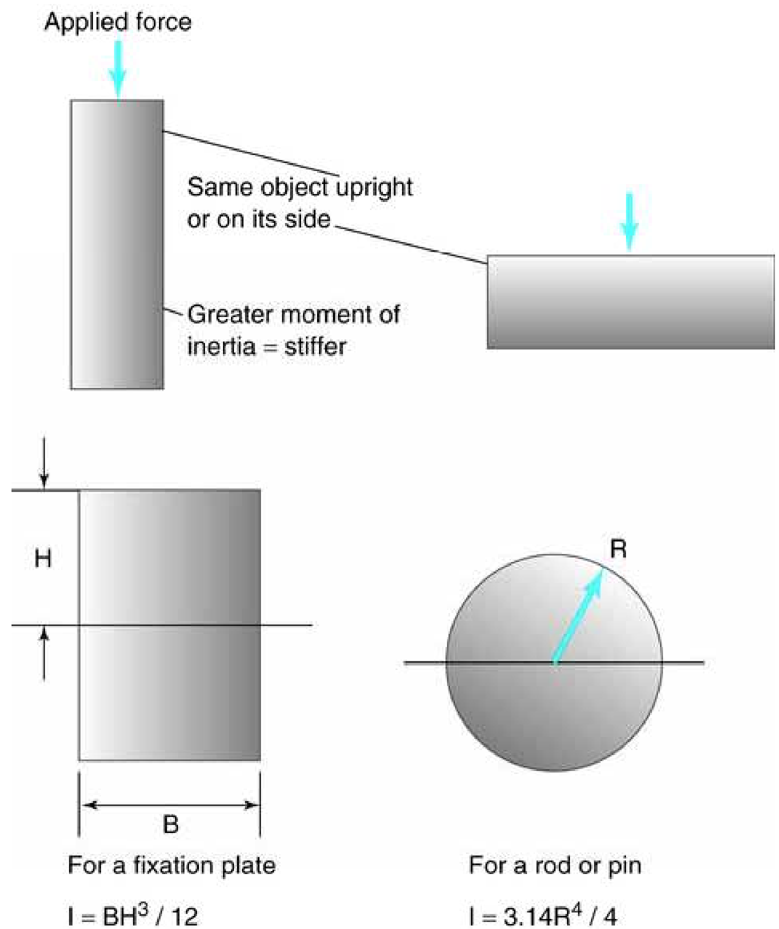


Figure from: Browner et al, Skeletal Trauma 2nd Ed, Saunders, 1998.

Implant Shape

- **Moment of Inertia**: further away material is spread in an object, greater the stiffness
- Stiffness and strength are proportional to ***radius^4^***



From: Justice

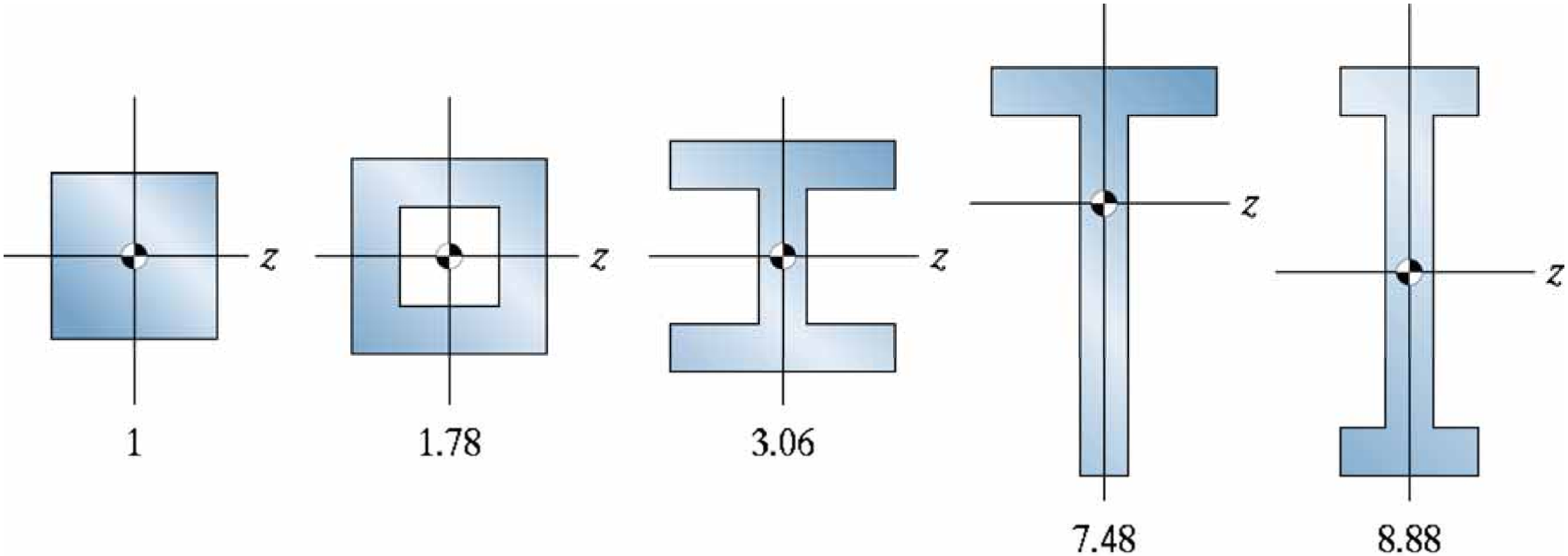
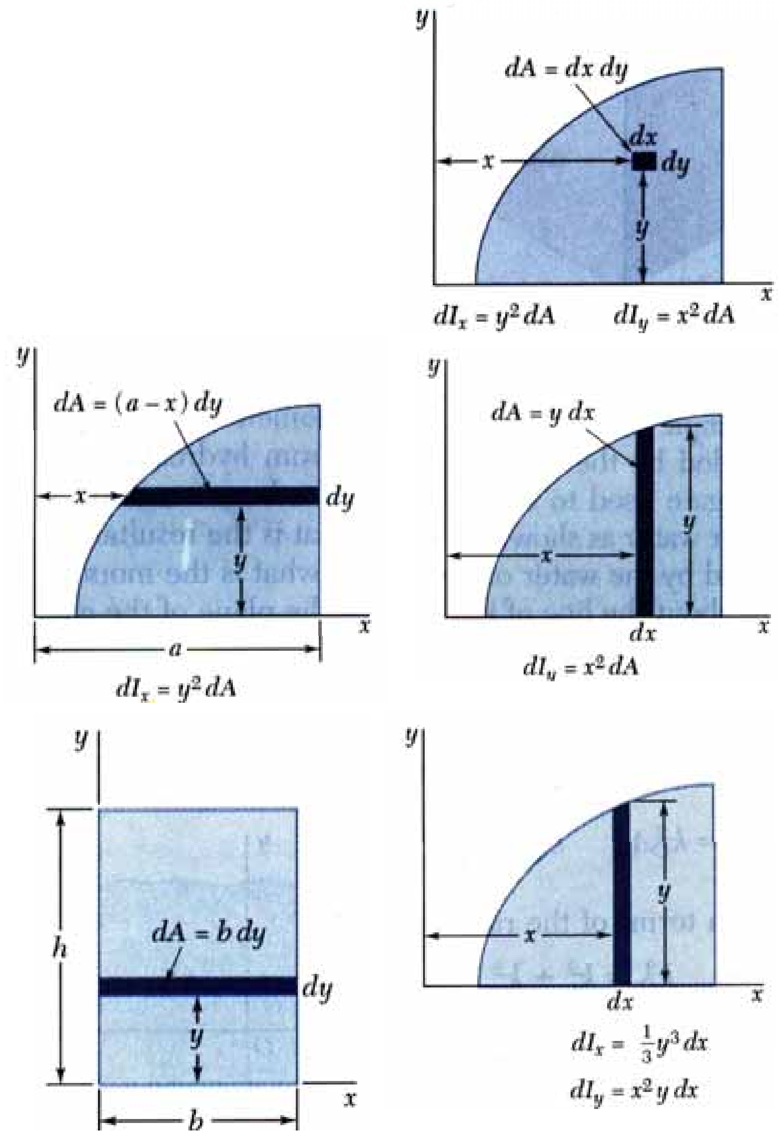


FIGURE 8-14 Typical beam cross sections and the ratio of *I* to the value for a solid square beam of equal cross-sectional area.

From: Hornsey

Moment of Inertia of an Area by Integration



- *Second moments* or *moments of inertia* of an area with respect to the x and y axes,

$$I_x = \int y^2 dA \qquad I_y = \int x^2 dA$$

- Evaluation of the integrals is simplified by choosing dA to be a thin strip parallel to one of the coordinate axes.

- For a rectangular area,

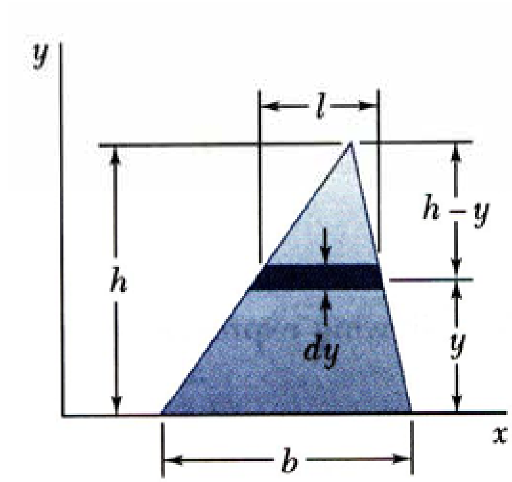
$$I_x = \int y^2 dA = \int_0^h y^2 b\,dy = \tfrac{1}{3} b h^3$$

- The formula for rectangular areas may also be applied to strips parallel to the axes,

$$dI_x = \tfrac{1}{3} y^3 dx \qquad dI_y = x^2 dA = x^2 y\,dx$$

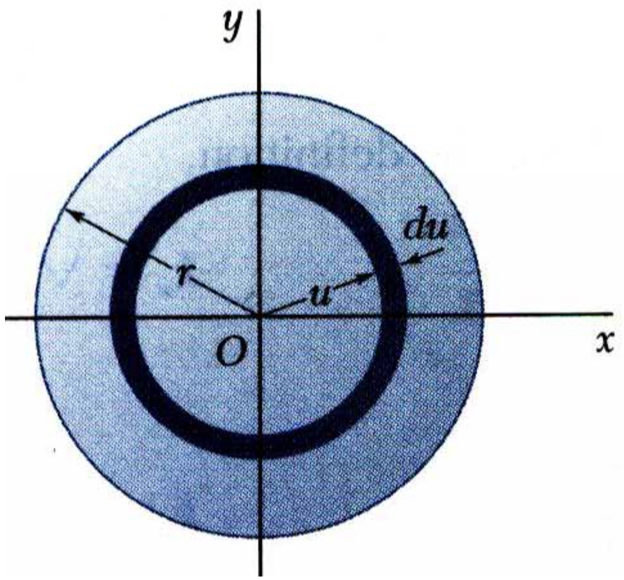
From: Rabiei

Homework Problem 16.1



Determine the moment of inertia of a triangle with respect to its base.

Homework Problem 16.2

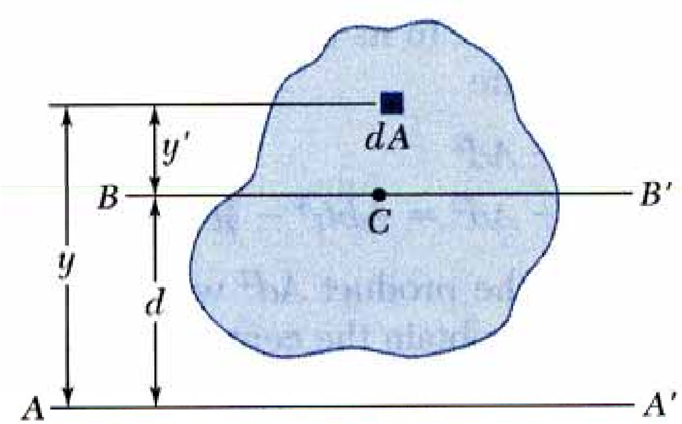


a) Determine the centroidal polar moment of inertia of a circular area by direct integration.

b) Using the result of part *a*, determine the moment of inertia of a circular area with respect to a diameter.

From: Rabiei

Parallel Axis Theorem



- Consider moment of inertia *I* of an area *A* with respect to the axis *AA'*

$$I = \int y^2 dA$$

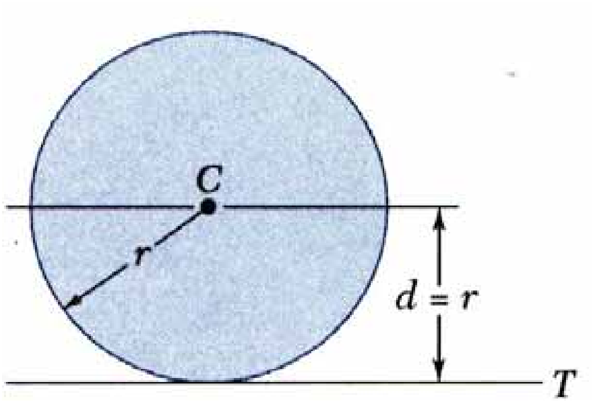
- The axis *BB'* passes through the area centroid and is called a *centroidal axis*.

$$I = \int y^2 dA = \int (y' + d)^2 dA$$
$$= \int y'^2 dA + 2d \int y' dA + d^2 \int dA$$

$$I = \bar{I} + Ad^2 \qquad \textit{parallel axis theorem}$$

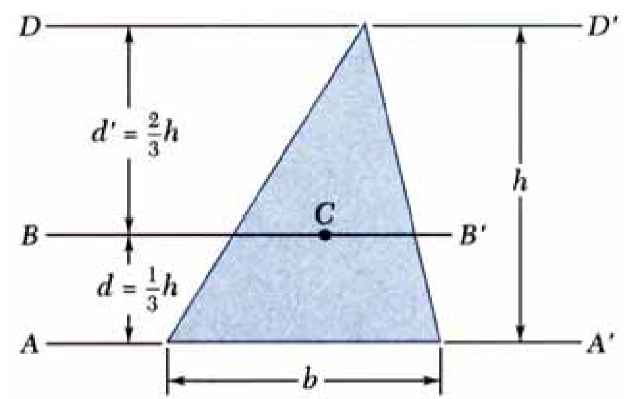
From: Rabiei

Parallel Axis Theorem



- Moment of inertia I_T of a circular area with respect to a tangent to the circle,

$$I_T = \bar{I} + Ad^2 = \tfrac{1}{4}\pi r^4 + \left(\pi r^2\right)r^2$$
$$= \tfrac{5}{4}\pi r^4$$



- Moment of inertia of a triangle with respect to a centroidal axis,

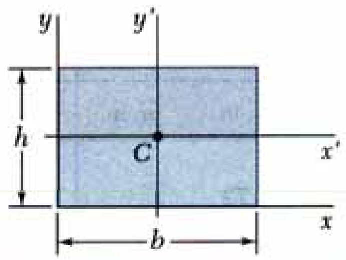
$$I_{AA'} = \bar{I}_{BB'} + Ad^2$$
$$I_{BB'} = I_{AA'} - Ad^2 = \tfrac{1}{12}bh^3 - \tfrac{1}{2}bh\left(\tfrac{1}{3}h\right)^2$$
$$= \tfrac{1}{36}bh^3$$

From: Rabiei

Moments of Inertia of Composite Areas

- The moment of inertia of a composite area A about a given axis is obtained by adding the moments of inertia of the component areas $A_1, A_2, A_3, \ldots$, with respect to the same axis.

Rectangle



$\bar{I}_{x'} = \frac{1}{12}bh^3$

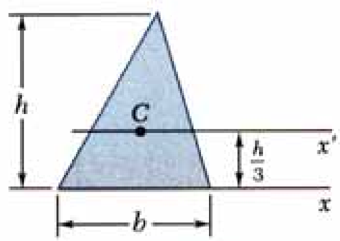
$\bar{I}_{y'} = \frac{1}{12}b^3h$

$I_x = \frac{1}{3}bh^3$

$I_y = \frac{1}{3}b^3h$

$J_C = \frac{1}{12}bh(b^2 + h^2)$

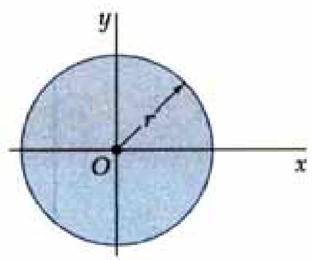
Triangle



$\bar{I}_{x'} = \frac{1}{36}bh^3$

$I_x = \frac{1}{12}bh^3$

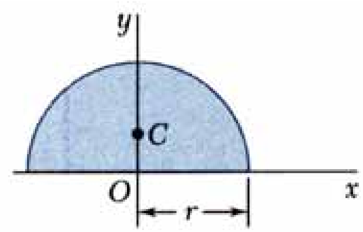
Circle



$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$

$J_O = \frac{1}{2}\pi r^4$

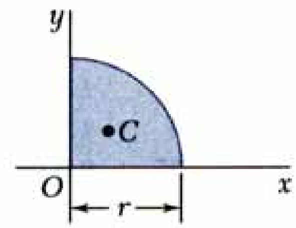
Semicircle



$I_x = I_y = \frac{1}{8}\pi r^4$

$J_O = \frac{1}{4}\pi r^4$

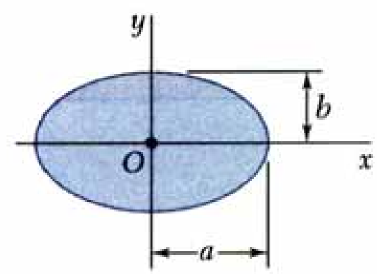
Quarter circle



$I_x = I_y = \frac{1}{16}\pi r^4$

$J_O = \frac{1}{8}\pi r^4$

Ellipse



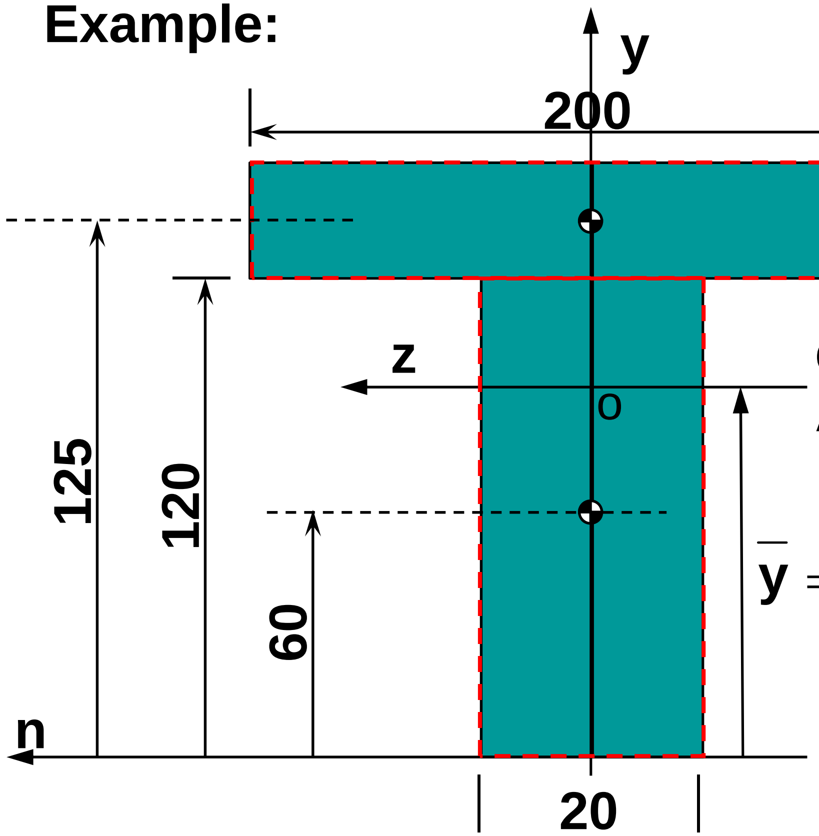
$\bar{I}_x = \frac{1}{4}\pi ab^3$

$\bar{I}_y = \frac{1}{4}\pi a^3b$

$J_O = \frac{1}{4}\pi ab(a^2 + b^2)$

From: Rabiei

Example: **(Dimensions in mm)**



$$\bar{y} = 89.6 \text{ mm}$$

$$\bar{y} = \frac{1}{A}\int_A y' \cdot dA$$

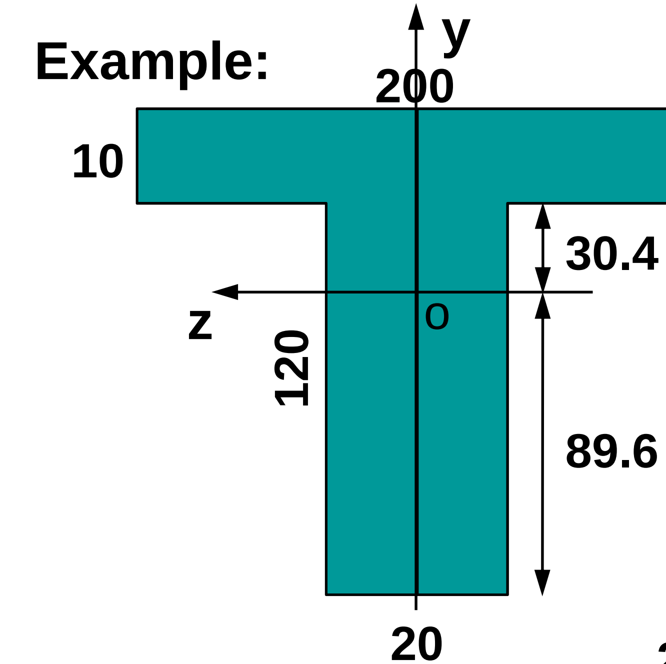
$$\bar{y} = \frac{1}{(200\times 10 + 120\times 20)}\left[(200\times 10)(125) + (120\times 20)(60)\right]$$

$$\bar{y} = \frac{1}{(4{,}400)}\left[250{,}000 + 144{,}000\right] = \frac{394{,}000}{4{,}400} = 89.55 \text{ mm}$$

$$= 89.6\times 10^{-3}\,\text{m}$$

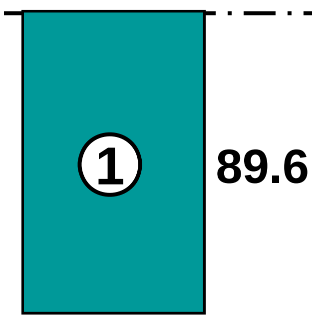
From: University of Auckland

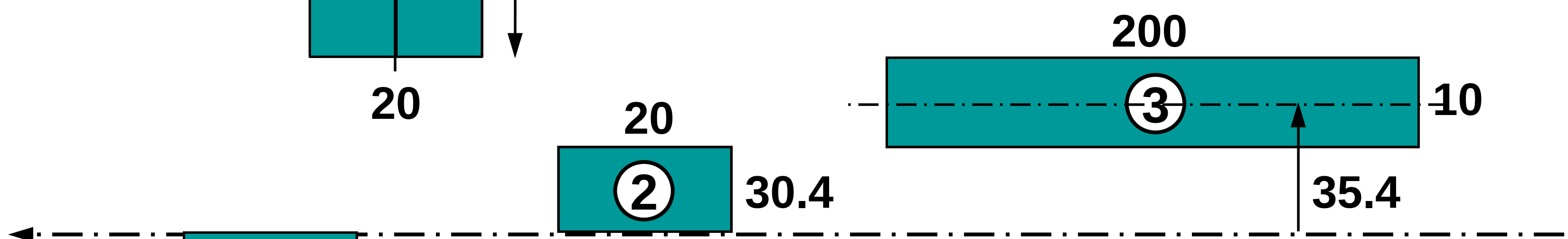
Example: (Dimensions in mm)



- **What is I_z?**
- **What is maximum σ_x?**

$$I_n = I_z + A\bar{y}^2$$



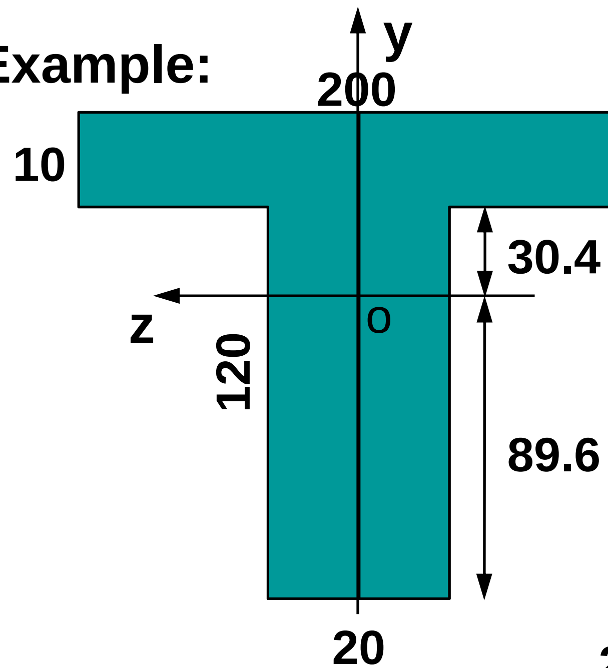


$$I_{z,1} = \frac{bd^3}{3} = \frac{(20)(89.6)^3}{3} = 4.79 \times 10^6 \text{ mm}^4$$

$$I_{z,2} = \frac{bd^3}{3} = \frac{(20)(30.4)^3}{3} = 0.19 \times 10^6 \text{ mm}^4$$

$$I_{z,3} = \frac{bd^3}{12} + A\bar{y}^2 = \frac{(200)(10)^3}{12} + (200 \times 10)(35.4)^2 = 3.28 \times 10^6 \text{ mm}^4$$

Example: **(Dimensions in mm)**



- **What is I_z?**
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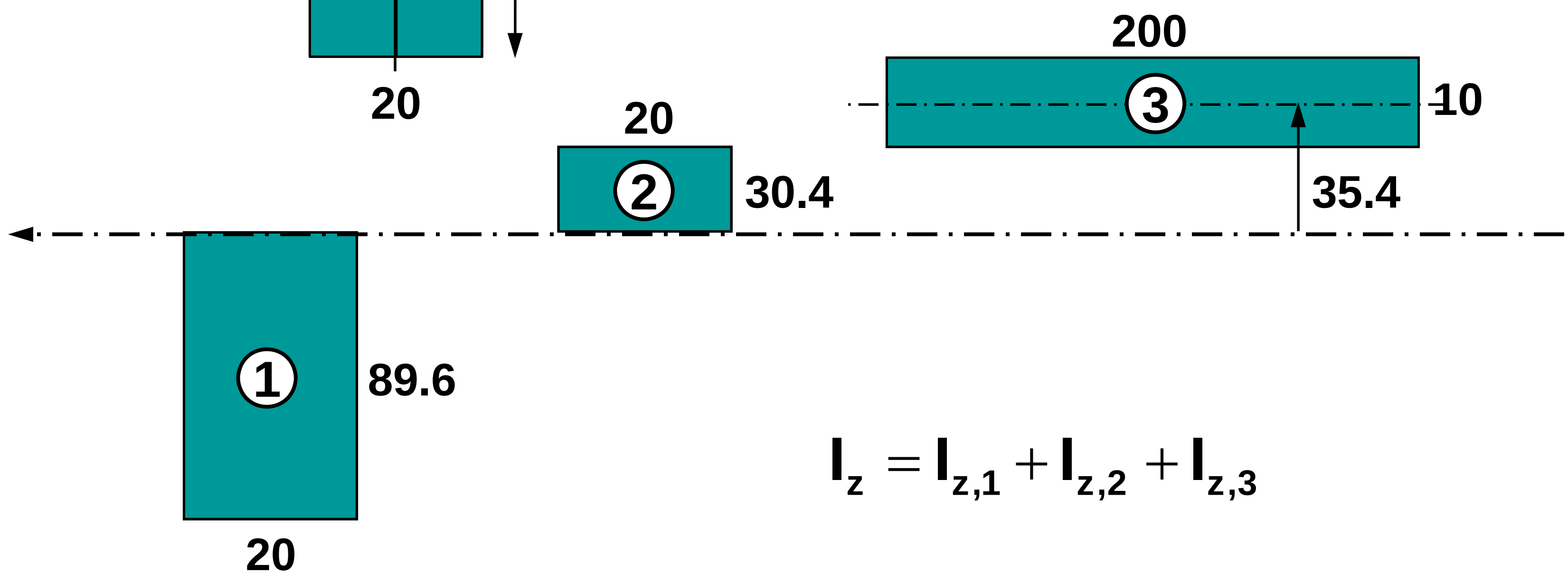
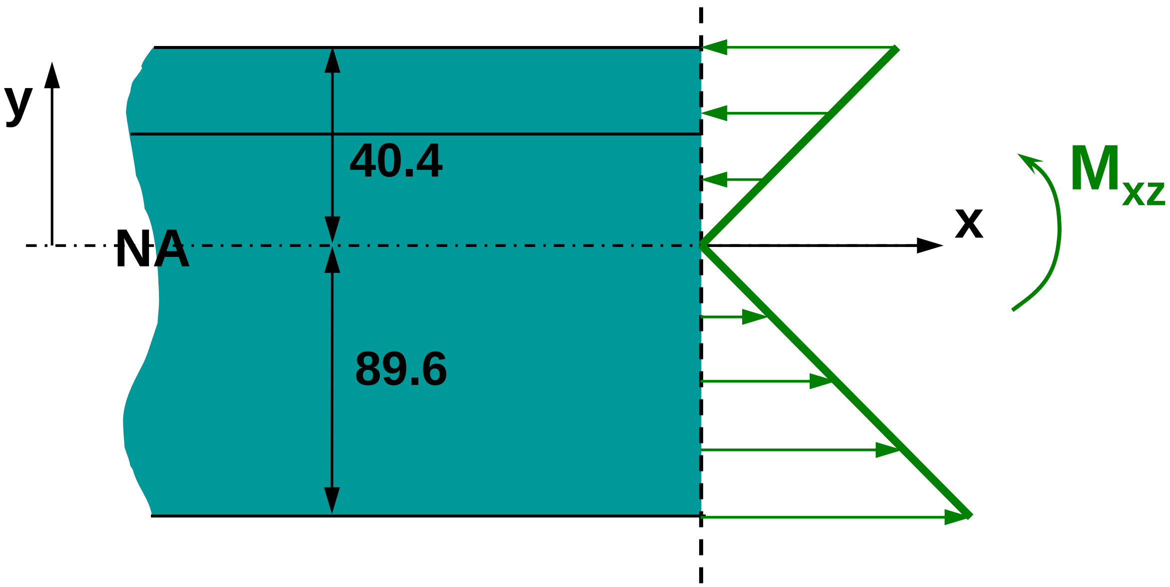


$$I_z = I_{z,1} + I_{z,2} + I_{z,3}$$

$$\Rightarrow I_z = 8.26 \times 10^6 \text{ mm}^4 = 8.26 \times 10^{-6} \text{ m}^4$$

Maximum Stress:

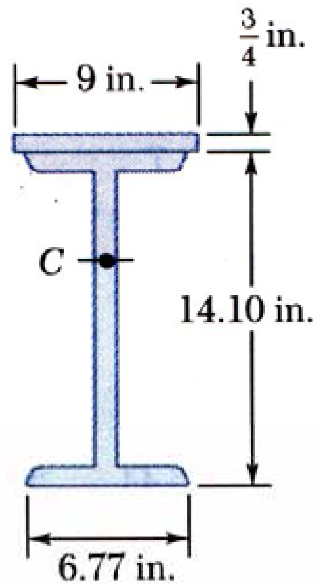


$$\sigma_x = -\frac{M_{xz}}{I_z} \cdot y'$$

$$\sigma_{x,Max} = -\frac{M_{xz}}{I_z} \cdot y_{Max}$$

$$\Rightarrow \sigma_{x,Max} = -\frac{M_{xz}}{\left(8.26 \times 10^{-6}\right)} \cdot \left(-89.6 \times 10^{-3}\right) \qquad \text{(N/m}^2 \textit{ or } \text{Pa)}$$

Homework Problem 16.3



The strength of a W14x38 rolled steel beam is increased by attaching a plate to its upper flange.

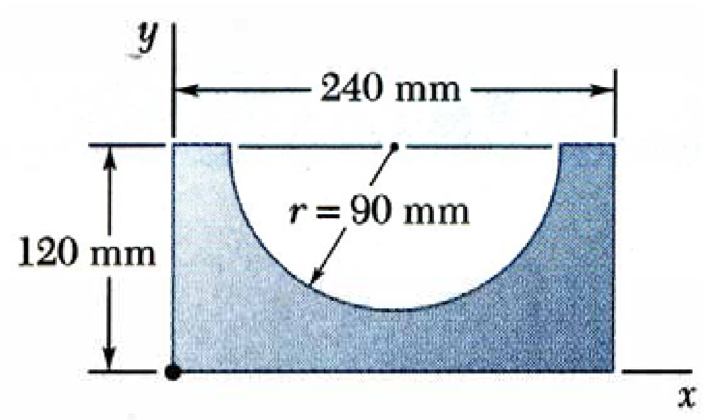
Determine the moment of inertia and radius of gyration with respect to an axis which is parallel to the plate and passes through the centroid of the section.

SOLUTION:

- Determine location of the centroid of composite section with respect to a coordinate system with origin at the centroid of the beam section.
- Apply the parallel axis theorem to determine moments of inertia of beam section and plate with respect to composite section centroidal axis.

From: Rabiei

Homework Problem 16.4



Determine the moment of inertia of the shaded area with respect to the *x* axis.

SOLUTION:

- Compute the moments of inertia of the bounding rectangle and half-circle with respect to the *x* axis.
- The moment of inertia of the shaded area is obtained by subtracting the moment of inertia of the half-circle from the moment of inertia of the rectangle.