Second Edition

ADVANCED MECHANICS OF MATERIALS

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From: Rapoff

CHAPTER 1

Orientation. Review of Elementary Mechanics of Materials

This chapter includes a selective and *brief* review of important assumptions, procedures, and results from a first course in mechanics of materials. Some items of importance are incorporated in subsequent chapters rather than appearing here. The reader is encouraged to consult a textbook of elementary mechanics of materials for detailed treatment of material reviewed in this chapter.

1.1 METHODS OF STRESS ANALYSIS

Typical questions posed in stress analysis are: Given the geometry of a body or structure, as well as its material properties, support conditions, and time-independent loads applied to it, what are the stresses and what are the displacements? A solution may be obtained by analytical, numerical, or experimental methods. Analytical methods include *mechanics of materials* and *theory of elasticity*. This book considers both, and places emphasis on the first.

Mechanics of materials is the engineer's way of doing stress analysis. The method involves the following steps.

1. Consider deformations produced by load, and establish (or approximate) how they are distributed over the body. This may be done by experiment, intuition, symmetry arguments, and/or prior knowledge of similar situations.
2. Analyze the geometry of deformation to determine how strains are distributed over a cross section.
3. Determine how stresses are distributed over a cross section by applying the stress-strain relation of the material to the strain distribution.
4. Relate stress to load. This step involves drawing a free-body diagram and writing equations of static equilibrium. The result is a formula for stress, typically in terms of applied loading and geometric parameters of the body.
5. Similarly, relate load to displacement, either by integration of the strain distribution determined in step 2 or by using energy arguments that relate work done by applied loads to elastic strain energy stored.

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Results of a mechanics of materials analysis may be exact, or good approximations, or rough estimates, depending mainly on the accuracy of assumptions made in the first step. Examples of the foregoing analysis are reviewed in subsequent sections, which point out that a substantial list of restrictions is needed if the resulting formulas are to be valid.

Theory of elasticity is the mathematician's way of doing stress analysis. In this method, one seeks stresses and displacements that simultaneously satisfy the requirements of equilibrium at every point, compatibility of all displacements, and boundary conditions on stress and displacement. In contrast to the mechanics of materials method, this method does not operate under any initial assumption or approximation about the geometry of deformation. Therefore theory of elasticity can solve a problem for which deformations cannot be reliably anticipated, such as the problem of determining stresses around a hole in a plate. However, the technique is more difficult than the mechanics of materials method and cannot be successfully applied to as great a variety of practical problems. Often, a practical problem is treated by a mixture of elasticity and mechanics of materials techniques.

Many problems of stress analysis are best solved numerically, on computers that range from PCs to supercomputers. Numerical analysis software is powerful and versatile; it has become comparatively easy to use and presents results graphically with great polish. None of this analytical power assures that results are even approximately correct. An analyst might easily blunder in deciding what simplifications are appropriate, in choosing the specific computational procedures to use, or in preparing input data. Computed results may contain large errors and, in any case, must be checked against results obtained in some other way. Mechanics of materials analysis serves well for checking, even in cases where it provides only a rough approximation. Regardless of the analysis method, success in solving a problem depends mainly upon the analyst's having clear insight into the phenomenon under study.

An analysis, by any method other than experiment, is applied to a model of reality rather than to reality itself. One cannot possibly take full account of the numerous details of the actual problem. Accordingly, the model is an idealization, in which geometry, loads, and/or support conditions are simplified, based on the analyst's understanding of which aspects of the actual problem are unimportant for the purpose at hand. Thus, a stress raiser may be temporarily neglected, weight of the body may be ignored, or a distributed load may be regarded as acting at a point. (As a practical matter, even the magnitude of loading is not usually known with much precision.) After devising a model, one must do all appropriate analyses. For example, one must not stop with stresses if buckling is also a possible mode of failure. Accordingly, a goal of studying stress analysis is to learn what idealizations and analysis goals are appropriate, which implies that one must learn how bodies of various shapes and support conditions respond to various loads.

Finally, some words about derivations. Why study the derivation of a formula? First, it makes the formula plausible. A more important reason is that a derivation makes clear the assumptions and restrictions needed in order to obtain the formula. Thus, by knowing the derivation, one can recognize situations in which a formula should *not* be applied.

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1.2 TERMINOLOGY

The following list is far from exhaustive. Terms listed are used throughout this book.

Beam: An elongated member, usually slender, intended to resist lateral loads by bending.

Body force: A loading that acts throughout a body rather than only on its surface. Self-weight and the inertia force of spinning about an axis are instances of body force.

Boundary conditions: Prescribed displacements at certain locations; for example, the stipulation that the supported end of a cantilever beam neither translates nor rotates. These boundary conditions may also be called *support conditions.* The term "boundary conditions" may also indicate prescribed stresses, forces, or moments. For example, at the unsupported end of a cantilever beam loaded only by its own weight, transverse shear force and bending moment must both vanish.

Brittle behavior: A material failure in which fracture surfaces show little or no evidence that failure has produced permanent deformation.

Cold working: Deformation that results in residual stresses. (In contrast, *hot working* is deformation at high enough temperature that stresses quickly dissipate by annealing.) Cold working by *shot peening* is the bombarding of an object by metal shot (roughly 0.2 mm to 4 mm in diameter) thrown at substantial velocity (roughly 70 m/s), the purpose being to produce residual compressive stresses in the surface layer.

Curvature: The reciprocal of the radius of curvature ρ, that is, $\kappa = 1/\rho$; used in beam theory.

Ductile behavior: Material behavior in which appreciable permanent deformation is possible without fracture.

Elastic: Material behavior in which deformations produced by load disappear when load is removed.

Elastic limit: The largest uniaxial normal stress for which material behavior is elastic. (Compare *yield strength.*)

Elastic modulus: The ratio of axial stress σ_a to axial strain ϵ_a in uniaxial loading; $E = \sigma_a/\epsilon_a$. Restricted to a linear relation between σ_a and ϵ_a. Also called *modulus of elasticity* or *Young's modulus.*

Fixed: A boundary condition in which all motion is prevented. Also called *built-in, clamped,* or *encastre.*

Flexure: Bending.

Frame: A structure built of bars, in which relative rotation between bars is prevented at joints, as by welding bars together where they meet. Bending of the bars is usually important in the calculation of stresses. (Compare *truss.*)

Homogeneous: Having the same material properties at all locations.

Isotropic: Having the same properties (stiffness, strength, conductivity, etc.) in every direction. As examples, glass is isotropic, wood is not isotropic. (Compare *orthotropic.*)

Lateral: Directed to the side; thus, directed normal to the axis of a beam or normal to the surface of a plate or a shell.

Nonlinear problem: A problem in which deflections or stresses are not directly proportional to the load that produces them. An example is the contact stress where a train wheel meets the rail. The area of contact grows as load increases. Another example is an initially flat membrane, like a trampoline. Lateral load is resisted by forces in the membrane that are functions of both the amount of deflection and the deflected shape.

Orthotropic: Having different stiffness (or other properties) in different directions, with the directions of maximum and minimum stiffness being mutually perpendicular. (Compare *isotropic.*)

Permanent set: Deformation that remains after removal of the load that produced it.

Plastic: A state of stress or deformation that results in permanent set if the load is removed.

Poisson's ratio: Designated by ν, where $\nu = -\epsilon_t/\epsilon_a$, and ϵ_t and ϵ_a are respectively the transverse and axial strains produced by a uniaxial stress σ_a below the proportional limit.

Principal stress: A normal stress σ, acting on an area A (or dA) when A (or dA) is free of shear stress. In this book, numerical subscripts on principal stresses indicate algebraic ordering, maximum to minimum; that is, $\sigma_1 \geq \sigma_2 \geq \sigma_3$.

Prismatic member: A straight bar with identical cross sections. In other words, a uniform straight member; the solid generated by translating a plane shape along a straight axis normal to its plane.

Proportional limit: The largest uniaxial normal stress for which stress is directly proportional to strain. (Compare *yield strength.*)

Safety factor (*SF*): The number by which the working load (the maximum load anticipated in normal service) must be multiplied to produce the design load (the load that causes failure). If the loading has more than one component force or moment, all components must change proportionally if this definition is to apply. If stress is the quantity indicative of failure, and if stress is directly proportional to applied load, then *SF* can also be regarded as the number by which the stress that causes the material to fail must be divided in order to obtain the allowable stress, which is the maximum stress to be allowed in service. Typically, design codes prescribe allowable stresses. The number chosen for *SF* is influenced by uncertainties about loads, material properties, quality of fabrication, and accuracy of design procedures; by the cost of failure; and by the cost of adopting a large *SF.*

Saint-Venant's principle: The proposition that two statically equivalent loadings, applied (separately) to the same region of a body, each produces essentially the same state of stress and deformation in the body at distances from the loaded region greater than the larger dimension of the loaded region. (*Caution:* This principle is not reliable for thin-walled construction or for some orthotropic materials.)

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Shaft: An elongated member, usually slender and straight, intended to resist torsional loads.

Shear modulus: The ratio of shear stress τ to shear strain γ; $G = \tau/\gamma$. Restricted to a linear relation between τ and γ. Also called *modulus of rigidity.*

Simply supported: A boundary condition in which lateral displacements are prevented but rotations are allowed. A simple support applies no moment to a structure. A simple support may also be called *pinned* or *hinged.*

Static indeterminacy: A condition in which one is unable to calculate all support reactions, or all internal forces or stresses, by use of only the conditions of static equilibrium. (Deformations must also be considered in order to obtain a complete solution.)

Static load: A load that does not vary with time. A more precise term would be "quasi-static load," because a truly static load could be neither applied nor removed.

Superposition: The principle that two or more static loads, applied sequentially in any order, produce the same final result as obtained by applying all loads simultaneously. The principle is not applicable in instances of nonlinearity of response, under either an individual load or combinations of loads.

Transverse: Across. Thus, for load or deflection, the same as *lateral.*

Truss: A structure built of bars in which each bar is idealized as a two-force member, as if ends of bars were connected together by frictionless pins. (Compare *frame.*)

Yield strength: The maximum uniaxial tensile stress that can be applied without exceeding a specified permanent set upon release of load. It may also be called *yield stress.* The specified permanent set is often taken as an axial strain of 0.002. In a metal, numerical values of the elastic limit, proportional limit, and yield strength are usually quite similar.

1.3 PROPERTIES OF A PLANE AREA

Properties of a plane area are often needed, particularly for beam problems. The more essential properties and manipulations are reviewed here.

Definitions. Consider a plane area A, with rectangular Cartesian coordinates st in the same plane, Fig. 1.3-1a. By definition,

$$I_s = \int_A t^2\,dA \qquad I_t = \int_A s^2\,dA \qquad I_{st} = \int_A st\,dA \tag{1.3-1}$$

I_s and I_t are *moments of inertia,* about s and t axes respectively. I_{st} is the *product of inertia.* I_s and I_t are always positive, but I_{st} can be positive, negative, or zero. Contributions $st\,dA$ are positive for areas dA in the first and third quadrants and negative for areas dA in the second and fourth quadrants (Fig. 1.3-1b). If s or t is a symmetry axis of A, then $I_{st} = 0$. The argument is shown in Fig. 1.3-1b. Each contribution $+st\,dA$ is matched by a contribution $-st\,dA$. Summing over A, we obtain $I_{st} = 0$.

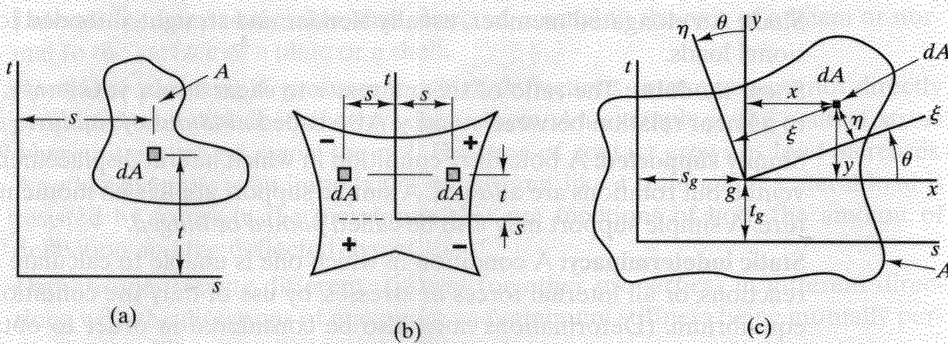

FIGURE 1.3-1 (a) Arbitrary plane area A. (b) Plane area symmetric about the t axis. Quadrants bear signs corresponding to their contribution to I_{st}. (c) Plane area with centroidal axes xy and $\xi\eta$.

Parallel Axis Theorems. These theorems relate quantities in Eq. 1.3-1 to corresponding quantities referred to *parallel* axes in the plane of A whose origin is at the *centroid of area* A. In Fig. 1.3-1c, let x and y be rectangular centroidal axes of A, respectively parallel to axes s and t, and in the plane of A. The parallel axis theorems are

$$I_s = I_x + At_g^2 \qquad I_t = I_y + As_g^2 \qquad I_{st} = I_{xy} + As_g t_g \tag{1.3-2}$$

where $I_x = \int y^2\,dA$, $I_y = \int x^2\,dA$, and $I_{xy} = \int xy\,dA$. Distances s_g and t_g are the coordinates of centroid g in the st system. These distances carry algebraic signs (both are positive in Fig. 1.3-1c). The argument for the last of Eqs. 1.3-2 is as follows. Substitute $s = s_g + x$ and $t = t_g + y$ into Eq. 1.3-1, and note that $\int x\,dA$ and $\int y\,dA$ both vanish because the xy system is centroidal. Thus

$$\begin{aligned} I_{st} &= \int_A (x + s_g)(y + t_g)dA = \int_A xy\,dA + 0 + 0 + s_g t_g \int_A dA \\ &= I_{xy} + As_g t_g \end{aligned} \tag{1.3-3}$$

The remaining two theorems in Eqs. 1.3-2 are proved in similar fashion.

Centroidal Principal Axes. In general, equations for principal axes do not require that axes be centroidal. However, in what follows, the origin of coordinates is placed at the centroid of area A because centroidal coordinates are the most useful.

Consider Fig. 1.3-1c. Systems xy and $\xi\eta$ are both rectangular, centroidal, and coplanar with A. The orientation of system xy can be chosen for convenience; for example, parallel to straight sides if area A happens to have them. System $\xi\eta$ is oriented at arbitrary angle θ with respect to system xy. Coordinates of a point in the rotated system $\xi\eta$ are $\xi = y \sin\theta + x\cos\theta$ and $\eta = y\cos\theta - x\sin\theta$. Thus we can obtain the following expressions by integration and substitution of trigonometric identities for $\sin^2\theta$, $\cos^2\theta$, and $\sin\theta\cos\theta$ (see Eqs. 1.10-1).

$$I_\xi = \int_A \eta^2\,dA \quad \text{yields} \quad I_\xi = \frac{1}{2}(I_x + I_y) + \frac{1}{2}(I_x - I_y)\cos 2\theta - I_{xy}\sin 2\theta \tag{1.3-4a}$$

$$I_{\xi\eta} = \int_A \xi\eta\, dA \qquad \text{yields} \qquad I_{\xi\eta} = \frac{1}{2}(I_x - I_y)\sin 2\theta + I_{xy}\cos 2\theta \quad (1.3\text{-}4b)$$

One can select θ so that the moment of inertia of A becomes a maximum about either the ξ axis or the η axis. If I_ξ is the maximum I, it happens that I_η is the minimum I, and vice versa. The maximum I and the minimum I are called *principal moments of inertia* and their corresponding axes are called *principal axes.* The value of θ that maximizes (or minimizes) I_ξ is called θ_p. It is determined from the equation $dI_\xi/d\theta = 0$, which yields

$$\tan 2\theta_p = \frac{2I_{xy}}{I_y - I_x} \quad (1.3\text{-}5)$$

Angle θ_p has two values, $\pi/2$ apart, one for I_{max}, the other for I_{min}. By using Eq. 1.3-5 in Eq. 1.3-4a, we obtain the *principal moments of inertia:*

$$I_{max,min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \quad (1.3\text{-}6)$$

Substitution of Eq. 1.3-5 into Eq. 1.3-4b yields $I_{\xi\eta} = 0$. That is, *the product of inertia is zero for principal axes.* The converse is also true: if $I_{\xi\eta} = 0$, then axes ξ and η are principal. Therefore, *if* ξ *or* η *is an axis of symmetry, then* ξ *and* η *are principal axes.*

From Eq. 1.3-6, we see that $I_{max} + I_{min} = I_x + I_y$. This relation can be useful in calculation, for example to determine I_{min} when I_{max}, I_x, and I_y have already been calculated. It may be physically obvious which of the two angles in Eq. 1.3-5 refers to the I_{max} axis, as in Fig. 1.3-2b. Otherwise the candidate angle can be substituted into Eq. 1.3-4a to see if I_ξ turns out to be I_{max} or I_{min}. Or, adapting a formula developed for stress transformation (see below Eq. 2.2-5), the counterclockwise angle θ_p from the x axis to the axis about which I is maximum is given by $\tan\theta_p = (I_x - I_{max})/I_{xy}$.

If $I_{max} = I_{min}$, angle θ does not matter. Then all centroidal axes yield the same I, and the product of inertia is zero for all these axes (Fig. 1.3-2c).

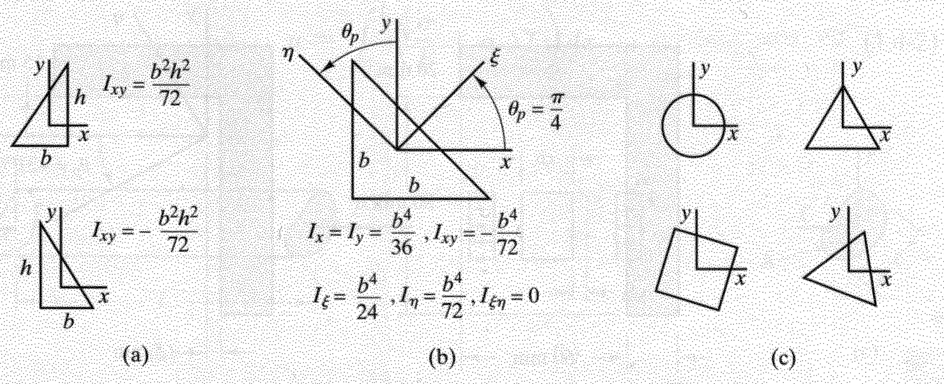

FIGURE 1.3-2 Various plane areas with centroidal axes xy. (a) Right triangles. (b) Isosceles right triangle. (c) Circle, square, and equilateral triangles. For each, $I_x = I_y$ and $I_{xy} = 0$.

Polar Moment of Inertia *J*. Let r be the distance from the origin of xy coordinates to an element of area dA. Then $r^2 = x^2 + y^2$, and with respect to the "pole" at $x = y = 0$,

$$J = \int_A r^2\,dA \quad \text{yields} \quad J = \int_A (x^2 + y^2)dA \quad \text{or} \quad J = I_y + I_x \tag{1.3-7}$$

The latter formula may be useful as a calculation device. Also, J is used in the torsional analysis of bars of circular cross section.

EXAMPLE

For the plane area in Fig. 1.3-3 we will determine I_x, I_y, I_{xy}, locate the principal centroidal axes, and determine the principal moments of inertia.

The centroid of A, at $x = y = 0$, has already been located, by means of calculations explained in textbooks about statics. For convenience in the following calculations, the cross section is arbitrarily divided into parts 1 and 2, as shown. Centroids of these parts are at $x = y = -15$ mm for part 1, and $x = y = 25$ mm for part 2. Equation 1.3-2 yields

$$I_x = \left[\frac{20(100)^3}{12} + 2000(-15)^2\right] + \left[\frac{60(20)^3}{12} + 1200(25)^2\right] \tag{1.3-8}$$

where the two bracketed expressions come from parts 1 and 2, respectively. I_y is obtained from a similar calculation and I_{xy} is

$$I_{xy} = [0 + 2000(-15)(-15)] + [0 + 1200(25)(25)] \tag{1.3-9}$$

Collecting results, we have

$$I_x = 2.907(10^6)\text{ mm}^4 \qquad I_y = 1.627(10^6)\text{ mm}^4 \qquad I_{xy} = 1.200(10^6)\text{ mm}^4 \tag{1.3-10}$$

From Eq. 1.3-5, we calculate the orientation of a principal axis.

$$\tan 2\theta_p = \frac{2(1.200)}{1.627 - 2.907} \quad \text{which yields} \quad \theta_p = -31.0^\circ \tag{1.3-11}$$

which is the clockwise angle shown in Fig. 1.3-3. The other possible angle, $\theta_p + 90^\circ$, is a 59° counterclockwise angle from the x axis to the η axis. From Eq. 1.3-6,

$$I_{\max} = 3.627(10^6)\text{ mm}^4 \qquad I_{\min} = 0.907(10^6)\text{ mm}^4 \tag{1.3-12}$$

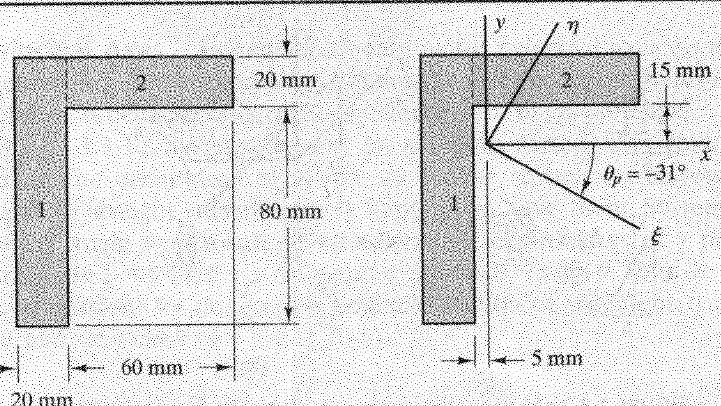

FIGURE 1.3-3 A plane area. Axes xy are centroidal. Axes $\xi\eta$ are centroidal and principal.

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In this example it is clear by inspection of Fig. 1.3-3 that $I_\xi = I_{\max}$ rather than $I_\eta = I_{\max}$. Angle $\theta_p = -31°$ to the $I_{\max}$ axis, shown in Fig. 1.3-3, is verified by the formula $\tan\theta_p = (I_x - I_{\max})/I_{xy}$.

1.4 AXIAL LOADING. PRESSURE VESSELS

Straight Bars. Consider a prismatic bar loaded by centroidal axial force P, Fig. 1.4-1a. The basic assumption about deformation is that plane cross sections remain plane when load P is applied. Thus any two cross sections a distance dx apart increase their separation an amount du (Fig. 1.4-1b), and axial strain is $\epsilon = du/dx$ at all points in a cross section. If the same stress-strain relation prevails throughout a cross section (that is, if the material is homogeneous), then axial stress σ is also the same at all points in a cross section. Equilibrium of axial forces requires that $\sigma A = P$. Thus the stress formula becomes $\sigma = P/A$. This result is not valid close to points of load application, where it is obvious that plane cross sections do not remain plane. According to Saint-Venant's principle, $\sigma = P/A$ should be an accurate formula at distances greater than ℓ from the loaded points, where ℓ is shown in Fig. 1.4-1c. The resultant force provided by a uniform stress distribution acts at the centroid of a cross section. For any cross section, load P must be collinear with this resultant. Therefore, if σ is to be uniformly distributed over a cross section, load P must be directed through centroids of cross sections. Accordingly, the bar cannot be curved. Taper, if not pronounced, causes little departure from the basic assumption; then σ is almost uniform over a cross section and is a function of axial coordinate x.

In uniaxial stress, a linearly elastic material has the stress-strain-temperature relation

$$\epsilon = \frac{\sigma}{E} + \alpha\,\Delta T \tag{1.4-1}$$

where α is the coefficient of thermal expansion and ΔT is the temperature change. From the strain expression $\epsilon = du/dx$, an increment of axial displacement is $du = \epsilon\,dx$. Combining this expression with Eq. 1.4-1 and integrating, we obtain

$$u = \int_0^L \left(\frac{\sigma}{E} + \alpha\,\Delta T\right) dx \tag{1.4-2}$$

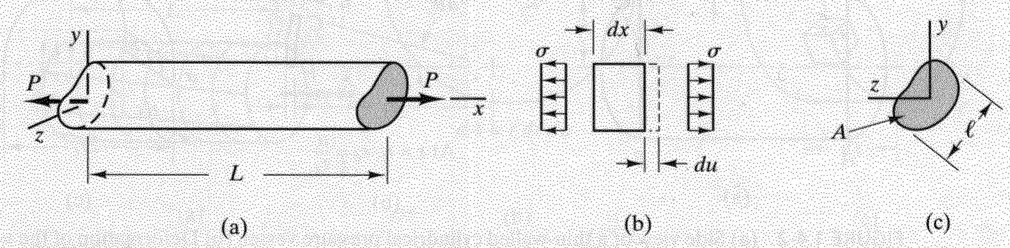

FIGURE 1.4-1 (a) Prismatic bar under centroidal axial load P. (b) Axial deformation, and axial stress σ. (c) Typical cross section.

as the axial deformation over a length L. (The symbol u is used in preference to δ or Δ in order to agree with notation in subsequent chapters, where u, v, and w denote displacement components in x, y, and z directions respectively.) Any of the quantities in parentheses in Eq. 1.4-2 may be a function of x. For the uniform bar in Fig. 1.4-1, with $\Delta T = 0$, Eq. 1.4-2 reduces to the familiar expression $u = PL/AE$. The presence of E in this formula—or in any other formula—makes it obvious that the formula is restricted to linearly elastic conditions.

Pressure Vessels. Let the cylindrical tank in Fig. 1.4-2 be thin walled, which customarily means that $r_i \approx 10t$ or more. Internal pressure causes points to displace radially but not circumferentially. Radial displacement u, greatly exaggerated, is shown in Fig. 1.4-2b. The initial length of arc CD is $r_i\,d\theta$. Its final length, after radial displacement u, is $(r_i + u)d\theta$. Its change in length is therefore $u\,d\theta$, and its circumferential strain is $\epsilon = (u\,d\theta)/(r_i\,d\theta) = u/r_i$. It is reasonable to assume that all points through the thickness have almost the same radial displacement u. Therefore, because all points also have almost the same radius, circumferential strain is almost uniform through the vessel wall. If the material is homogeneous, uniform strain implies uniform stress. Hence, summing forces in the direction of pressure p in Fig. 1.4-2c, we obtain

$$p(2r_i\,dx) = 2(\sigma t\,dx) \qquad \text{from which} \qquad \sigma = \frac{pr_i}{t} \tag{1.4-3}$$

In similar fashion one can obtain axial stress $pr_i/2t$ in the cylindrical tank and stress $pr_i/2t$ in any surface-tangent direction in a spherical tank. These formulas are not reliable, even for thin-walled pressure vessels, near changes in geometry such as AA and BB in Fig. 1.4-2a, which are circles where end caps are connected to the cylindrical vessel.

If the vessel were thick walled, we could not conclude that circumferential strains are almost uniform through the vessel wall. Imagine, for example, that $t = r_i$. Then, for circumferential strain to be the same both inside and outside, radial displacement of the outer surface would have to be twice that of the inner surface. This conclusion is unreasonable. In fact, the inside displaces somewhat more than the outside. Thus, if the wall is thick, the inner surface carries higher strain and therefore higher stress than the outer surface. Considerations from theory of elasticity are needed to obtain expressions for stresses in a thick-walled cylinder under internal pressure.

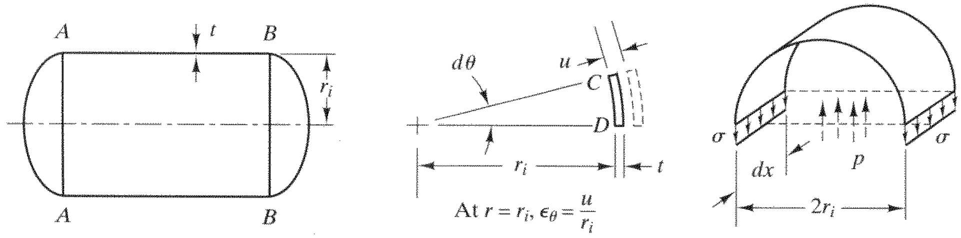

FIGURE 1.4-2 (a) Side view of a thin-walled cylindrical pressure vessel. (b) Deformation of the vessel wall due to internal pressure, viewed axially. (c) Circumferential stress, exposed by a cutting plane that contains the axis of the cylinder.

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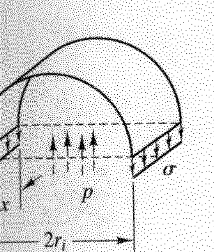

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1.5 TORSION

Consider a prismatic bar of circular cross section, whose material is homogeneous and isotropic. The geometry of deformation, which may be established by experiment or by symmetry arguments, is that initially plane cross sections remain plane when the bar is twisted. Also, radial straight lines remain straight, and rotate about the axis. The diameter and length of the bar do not change. From all this one deduces that radial, circumferential, and axial normal strains are absent, and that shear strain γ varies linearly with distance r from the axis but is independent of the circumferential and axial coordinates. If a rectangular grid is drawn on the surface of the bar, one finds that twisting produces the deformed grid shown in Fig. 1.5-1a. All right angles of the grid change by the same amount. This amount is the value of shear strain γ at radius $r = c$.

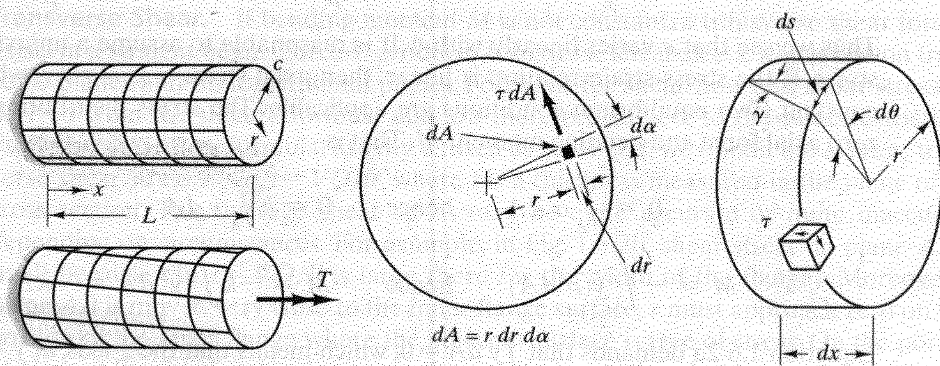
Let the shear stress versus shear strain relation be linear, $\tau = G\gamma$. Then, since shear strain γ varies linearly with distance from the axis, so does shear stress τ: symbolically, $\tau = kr$, where k is a constant. To relate τ to the torque T that produces it, we consider equilibrium of moments about the axis of the bar. Thus, from Fig. 1.5-1b.

$$T = \int_A r(\tau\, dA) \qquad \text{or} \qquad T = k\int_A r^2\, dA = kJ \tag{1.5-1}$$

Hence $k = T/J$, and the expression $\tau = kr$ becomes $\tau = Tr/J$, which is the standard torsion formula. Note that τ acts on longitudinal planes as well as on transverse planes, as shown in Fig. 1.5-1c.

Figure 1.5-1c leads to a formula for θ, the angle of twist of one end of the bar relative to the other. Angles γ and $d\theta$ are small, so

$$ds = \gamma\, dx = r\, d\theta \qquad \text{hence} \qquad \theta = \int_0^L \frac{\gamma}{r}\, dx \tag{1.5-2}$$



(a) (b) (c)

FIGURE 1.5-1 (a) Deformation produced by torque T applied to a bar of circular cross section. (b) Force increment $\tau\, dA$ produces torque increment $r(\tau dA)$. (c) Geometry of deformation that leads to a formula for angle of twist θ.

This result does not require that the material be linearly elastic. But if it is, we can substitute $\gamma = \tau/G = Tr/GJ$, whereupon the integrand in Eq. 1.5-2 becomes $T\,dx/GJ$. If T, G, and J are independent of x, we obtain the familiar expression $\theta = TL/GJ$. The presence of G in this formula makes it obvious that the formula is limited to linearly elastic conditions.

The manner of support or torsional load application, or the presence of stress raisers such as circumferential grooves, causes only local disturbances of stress, in accord with Saint-Venant's principle. These disturbances have little effect on the angle of twist. If the bar is tapered, then $J = J(x)$. The formula $\tau = Tr/J$ has little error provided the taper is slight. Changes that would invalidate our simple formulas, and the reasons why, are as follows. Orthotropy, unless it is polar about the axis of the bar, would make γ and τ depend on the circumferential coordinate as well as on r. The same effect would be produced by material properties that vary circumferentially, and by a noncircular cross section (see Section 7.11). A sharply curved geometry, as for the coil of a massive helical spring, would make γ larger toward the inside of the coil (see Section 6.1).

1.6 BEAM STRESSES

Bending. Consider a prismatic beam, whose material is homogeneous and isotropic. We require that the beam have a plane of symmetry, and that the beam be bent to an arc in this plane (Fig. 1.6-1). The geometry of deformation can be established by experiment or by symmetry arguments: Initially plane cross sections remain plane when bending moment is applied. Arbitrary cross sections AB and CD have the relative rotation $d\theta$. At coordinate y, axial strain is $-\epsilon$ and axial displacement is $-\epsilon\,dx$, negative because ϵ is compressive when y is positive. With ρ the radius of curvature, the small angle $d\theta$ can be expressed in two ways.

$$d\theta = \frac{-\epsilon\,dx}{y} \qquad \text{and} \qquad d\theta = \frac{dx}{\rho} \qquad \text{hence} \qquad \epsilon = -\frac{y}{\rho} \tag{1.6-1}$$

Thus we see that ϵ varies linearly with y. It is reasonable to assume a uniaxial state of stress. If the stress-strain relation is linear, then axial stress σ is $\sigma = ky$, where k is a constant. Two equilibrium conditions are applicable: The stress distribution provides zero axial force and bending moment M. That is,

$$0 = \int_A \sigma\,dA \qquad \text{hence} \qquad 0 = k\int_A y\,dA \tag{1.6-2a}$$

$$M = -\int_A y(\sigma\,dA) \qquad \text{hence} \qquad M = -k\int_A y^2\,dA = -kI \tag{1.6-2b}$$

Equation 1.6-2a demands that $\int y\,dA = 0$, which means that the z axis, at $y = 0$, passes through the centroid of the cross section. From Eq. 1.6-2b we obtain $k = -M/I$, where I is the moment of inertia of cross-sectional area A about its centroidal axis z. Hence the expression $\sigma = ky$ becomes $\sigma = -My/I$, which is the standard flexure formula. Typically we write simply $\sigma = My/I$, because the algebraic sign of σ at a given y is obvious from the direction of the bending moment.

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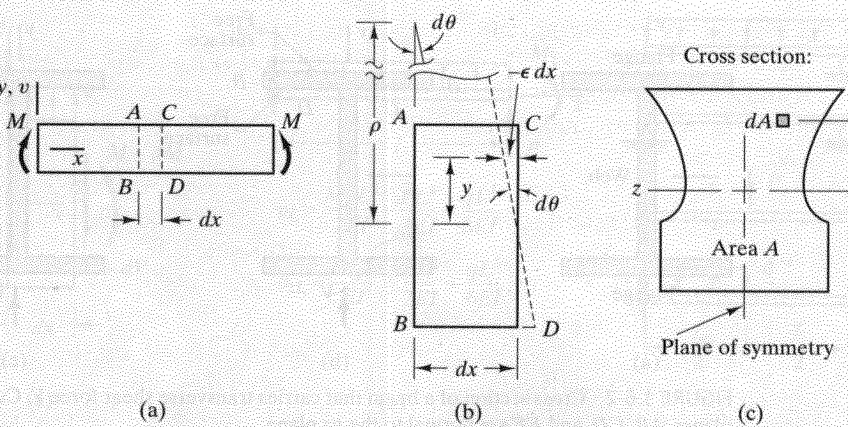

FIGURE 1.6-1 (a) Beam bent in the xy plane. (b) Deformations in the xy plane. (c) Arbitrary (but symmetric) cross section of area A. The symmetry plane of the cross section is normal to the paper.

Common situations to which the flexure formula is not applicable, or applicable only after modification, are as follow. If there is no symmetry plane, we cannot presume that axial strain ϵ is independent of z (axis z is shown in Fig. 1.6-1c). That is, Eq. 1.6-1 is no longer correct. This situation is called unsymmetric bending and is discussed in Chapter 10. If the beam has pronounced initial curvature before load is applied, plane cross sections still remain plane, but we cannot conclude that ϵ varies linearly with y (see Chapter 6). If the material is not linearly elastic, then $\sigma \neq ky$, and the latter forms of Eqs. 1.6-2 no longer apply. Similarly, if the material is not homogeneous, then $\sigma \neq ky$. Therefore we cannot use $\sigma = My/I$ to analyze a reinforced concrete beam. Finally, if the cross section is wide we must consider that the body is a plate rather than a beam (Chapter 12).

Transverse Shear. If bending moment M is not constant, a transverse shear force V exists in a straight beam. Force V produces transverse shear stress, which acts on transverse planes and on longitudinal planes. Formulas for shear flow and shear stress are derived from the flexure formula and are therefore subject to the same restrictions. From the shear flow formula, usually written as $q = VQ/I$, we obtain the *average* transverse shear stress $\tau = q/t = VQ/It$, where t is a thickness measured in the plane of the cross section. This average shear stress may be quite accurate or quite inaccurate, depending on circumstances. For example, in Fig. 1.6-2b, shear stress on plane AB is small because t in $\tau = VQ/It$ is large (here t is the width of the flange). Moreover, if plane AB is moved very close to the inner flange surface, τ must approach zero on that portion of the inner flange where the adjacent surface is free of stress. On the portion of plane AB immediately adjacent to the web, τ approaches the transverse shear stress on plane CD, where t is the web thickness and $\tau = VQ/It$ is accurate. The largest transverse shear stress in the flange is exposed by a vertical cutting plane such as EF, where t in VQ/It is the flange thickness. Details of these matters, and of how to use the formula VQ/It, appear in textbooks of elementary mechanics of materials.

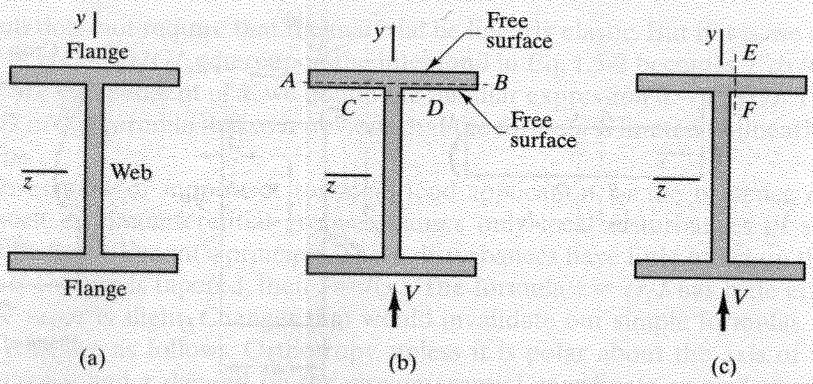

FIGURE 1.6-2 Cross section of a beam that carries transverse shear force V. Cutting planes AB, CD, and EF are normal to the yz plane.

1.7 BEAM DEFLECTIONS

Briefly, the formula that relates lateral deflection v to bending moment M is developed as follows. We use notation in Fig. 1.6-1. If $|dv/dx| \ll 1$, as is usual in practical beams, then the curvature of the deformed beam can be written as $1/\rho = d^2v/dx^2$. Also, for a linearly elastic material, Eq. 1.6-1 and the flexure formula $\sigma = -My/I$ yield another expression for curvature: $1/\rho = -\epsilon/y = -(\sigma/E)/y = -(-My/EI)/y = M/EI$. Equating the two expressions for curvature, we obtain

$$\frac{d^2v}{dx^2} = \frac{M}{EI} \tag{1.7-1}$$

Restrictions on this formula include those on the flexure formula. Also, deflections must be sufficiently small that slope $\theta = dv/dx$ of the deformed beam is everywhere much less than unity in magnitude. Transverse shear deformation has been neglected. Equation 1.7-1 actually says that M/EI is equal to the *change* in curvature. This viewpoint may become important for a beam having initial curvature before load is applied. For a beam initially straight and then bent to radius ρ, the initial curvature is zero, and the change in curvature is $(1/\rho - 0) = 1/\rho$.

An alternative form of Eq. 1.7-1 can be written, as follows. Equations of static equilibrium, applied to Fig. 1.7-1a, yield $dM/dx = V$ and $dV/dx = q$, where q is the intensity per unit length of distributed lateral load. Hence $d^2M/dx^2 = q$. For M we can substitute $EI(d^2v/dx^2)$ from Eq. 1.7-1. Thus

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = q \quad \text{or} \quad EI\frac{d^4v}{dx^4} = q \quad \text{if } EI \text{ is independent of } x \tag{1.7-2}$$

The latter form will be useful in subsequent chapters.

One can determine beam deflections (or solve statically indeterminate beam problems) by integrating Eq. 1.7-1 and making use of support conditions to evaluate constants of integration (and redundant reactions). Usually it is easier to solve these problems by use of tabulated beam formulas and the superposition principle. Indeed,

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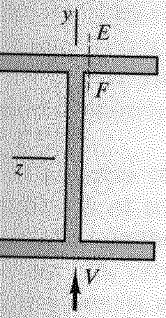

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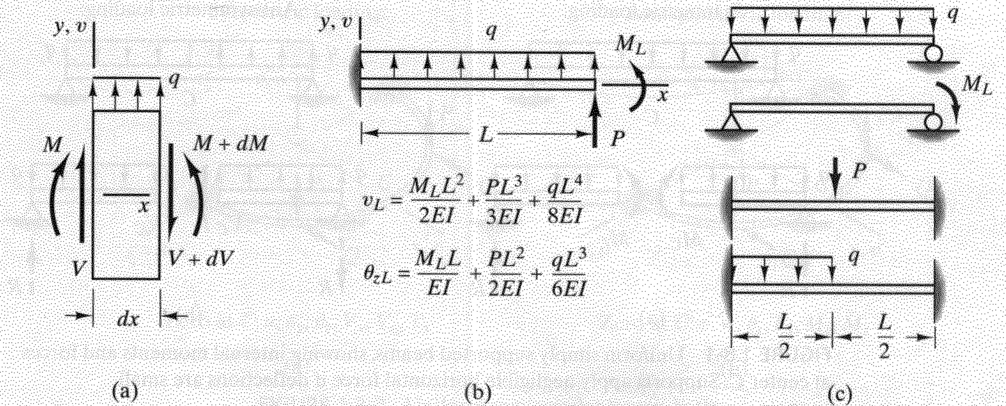

FIGURE 1.7-1 (a) Loads on a differential element of a beam. (b) Formulas for tip deflection and tip rotation of a uniform cantilever beam. (c) Problems of deflection, rotation, or static indeterminacy solvable by use of formulas in (b).

the few formulas in Fig. 1.7-1b are sufficient to solve most common problems of straight beams, including all those shown in Fig. 1.7-1c. An example problem is solved in Section 1.8. Like Eq. 1.7-1, formulas in Fig. 1.7-1b require that $|\theta| \ll 1$ throughout the beam.

1.8 SYMMETRY CONSIDERATIONS. STATIC INDETERMINACY

Symmetry Considerations. Sometimes one can exploit symmetry to obtain internal forces, determine support conditions, or reduce the effort required for analysis. For example, consider the simply supported beams in Fig. 1.8-1. Both have symmetry of geometry, elastic properties, and support conditions with respect to a plane normal to the beam axis at its center. The beams differ only in loading. In Fig. 1.8-1a, a mirror reflection of either half in the symmetry plane yields the other half in geometry, elastic properties, loading, support reactions, deformations, and internal forces at the symmetry plane. For antisymmetric loading, Fig. 1.8-1b, one half yields the other half after reflection and *reversal* of loading, support reactions, deformations, and internal forces at the symmetry plane. These considerations, in combination with the action-reaction nature of internal forces exposed by cutting open the beam, preclude the existence of shear forces V_C for symmetric loading and bending moments M_C for antisymmetric loading. Thus in either case the number of unknowns is immediately reduced by half.

The same considerations can be used in three dimensions. The semicircular beams in Fig. 1.8-2a lie in the xy plane. For each, there is symmetry of geometry, elastic properties, and support conditions about the yz plane. For symmetric loading, Fig. 1.8-2a, symmetry considerations dictate that at midpoint C there is no x direction displacement, no rotation about the y axis or the z axis, no transverse shear force in the y direction or the z direction, and no torque about the x axis. These conditions are listed in Fig. 1.8-2a. Unknowns at C are displacements v and w, rotation θ_x about the x axis, axial force F_x, and bending moments M_y and M_z about the y and z axes. These unknowns could be

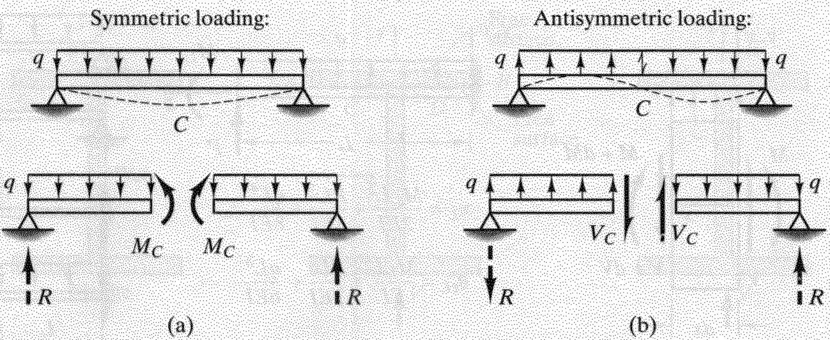

FIGURE 1.8-1 Uniform simply supported beams, showing internal moments and forces at center C. Supports apply negligible horizontal force if deflections are small.

determined by analysis of either half of the semicircular beam. In Fig. 1.8-2b two of the forces P and Q are reversed, so the load is antisymmetric. Symmetry considerations dictate the zero quantities listed in Fig. 1.8-2b. Again, analysis of either half of the beam is sufficient to determine the unknown quantities at C, which are $u, \theta_y, \theta_z, V_y, V_z$, and T_x.

The foregoing arguments are not immediately obvious. The reader is urged to consider these examples patiently, and to make supplementary sketches that show internal forces and moments.

Static Indeterminacy. The term is defined in Section 1.2. Calculations are illustrated by the following examples.

The stepped bar in Fig. 1.8-3a is all of the same material. It is to be uniformly heated from its stress-free temperature while confined between rigid walls. Statics tells us only that the walls apply forces P of equal magnitude. To determine them we must use a compatibility condition, which here is that the bar has no net change in length from end to end. Thus, taking P positive in tension and presuming that conditions are linearly elastic, we write

$$\alpha L\,\Delta T + \alpha L\,\Delta T + \frac{PL}{AE} + \frac{PL}{(2A)E} = 0 \tag{1.8-1}$$

where α is the coefficient of thermal expansion and ΔT is the temperature change. Solving for P and then for stresses $\sigma_1 = P/A$ and $\sigma_2 = P/2A$, we obtain

$$P = -\frac{4EA\alpha\,\Delta T}{3} \qquad \sigma_1 = -\frac{4E\alpha\,\Delta T}{3} \qquad \sigma_2 = -\frac{2E\alpha\,\Delta T}{3} \tag{1.8-2}$$

Note that axial strains are not zero, even though the overall change in length is zero. For example, in part 1, $\epsilon_1 = (\sigma/E) + \alpha\,\Delta T = -\alpha\,\Delta T/3$. Note also that modest temperature change can produce large stress. In the present example, if the bar is steel and $\Delta T = 100°\text{C}$, then σ_1 is about 320 MPa in magnitude.

As a second example, consider the beam in Fig. 1.8-3b. It is statically indeterminate to the second degree. Symmetry considerations can be used to reduce the degree of indeterminacy. Imagine that M_C is applied as two couples $M_C/2$, an infinitesimal dis-

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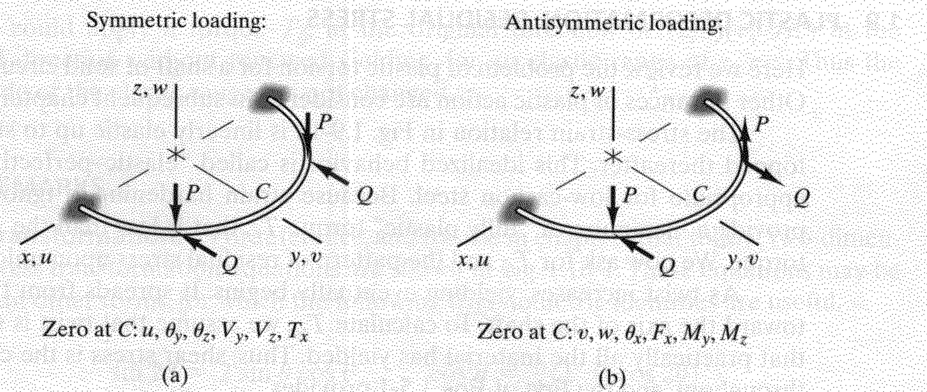

Zero at C: $u, \theta_y, \theta_z, V_y, V_z, T_x$ (a)

Zero at C: $v, w, \theta_x, F_x, M_y, M_z$ (b)

FIGURE 1.8-2 Uniform semicircular beams in the xy plane.

tance apart, and straddling point C. The loading is antisymmetric, so at C there is a transverse shear force V_C but zero bending moment and zero vertical displacement (Fig. 1.8-3c). Using formulas in Fig. 1.7-1b to state that the transverse displacement is zero at C, we solve for V_C and then for moment M_B at the wall.

$$-\frac{(M_C/2)a^2}{2EI} + \frac{V_C a^3}{3EI} = 0 \qquad \begin{cases} V_C = \dfrac{3M_C}{4a} \\ M_B = V_C a - \dfrac{M_C}{2} = \dfrac{M_C}{4} \end{cases} \tag{1.8-3}$$

Finally, having resolved the indeterminacy, we can use Fig. 1.7-1b again to determine the rotation at C.

$$\theta_C = \frac{(M_C/2)a}{EI} - \frac{V_C a^2}{2EI} \quad \text{hence} \quad \theta_C = \frac{M_C a}{8EI} \tag{1.8-4}$$

Problems such as those in Fig. 1.8-3 are probably called to mind by the term "statically indeterminate analysis." However, the term is also appropriate for the derivation of conventional stress formulas such as $\sigma = My/I$: an equilibrium equation, such as the first of Eqs. 1.6-2b, yields the second only when it is known how stress varies over the cross section. The variation is obtained by consideration of displacements.

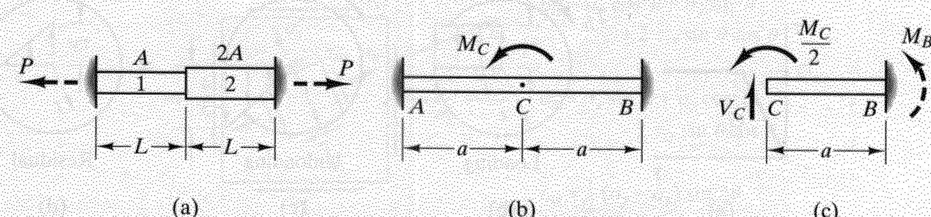

FIGURE 1.8-3 (a) Stepped bar held by rigid walls. (b) Statically indeterminate beam. (c) Right half of the beam, with symmetry considerations exploited.

1.9 PLASTIC DEFORMATION. RESIDUAL STRESS

Here we review the problem of plastic torsion for a shaft of solid circular cross section. Other instances of plastic action are considered in subsequent chapters.

The stress-strain relation in Fig. 1.9-1a is linearly elastic up to stress τ_Y and flat-topped thereafter. This idealized behavior is called "elastic–perfectly plastic" and is appropriate for low-carbon steel. Because strain hardening is ignored, calculations provide a maximum or "fully plastic" torque T_{fp} that is less than the actual maximum torque. We now ask for T_{fp} and the pattern of residual stress upon unloading.

As twist increases, yielding eventually begins. It spreads from the outer surface toward the axis of the shaft. To calculate T_{fp}, we assume that twist is sufficiently great that practically all the material has yielded. Thus, shear stress is the constant value τ_Y throughout, and the first of Eqs. 1.5-1 provides

$$T_{fp} = \tau_Y \int_0^{2\pi} \int_0^c r(r\,dr\,d\alpha) = \tau_Y \frac{2\pi c^3}{3} \qquad \text{hence} \qquad T_{fp} = \frac{4}{3}\left(\tau_Y \frac{\pi c^3}{2}\right) \tag{1.9-1}$$

where the latter expression in parentheses is the torque that initiates yielding, obtained from the torsion formula for linearly elastic conditions; that is, $\tau = Tr/J$ with $\tau = \tau_Y$ at $r = c$. This result shows that torque can be increased 33% after yielding begins.

Unloading can be accomplished by superposing on T_{fp} a torque of equal magnitude but reversed in direction. Anticipating that unloading will be elastic, we obtain the stress distribution in Fig. 1.9-1c from the reversed torque $T = T_{fp}$ and the elastic stress formula $\tau = Tr/J$. At first glance this calculation may appear wrong because the largest stress exceeds τ_Y. However, stresses in Fig. 1.9-1c always appear in *combination* with stresses in Fig. 1.9-1b. In combination, τ never exceeds τ_Y in magnitude, so unloading does not produce further yielding. If torque T_{fp} is again applied, residual stresses combine with the reverse of stresses in Fig. 1.9-1c to produce again the fully plastic stress pattern of Fig. 1.9-1b, but without renewed yielding.

The residual angle of twist after unloading cannot be calculated because we have not specified how much the shaft was twisted in producing T_{fp}. An *infinite* angle of twist would be required to bring inelastic strains all the way to $r = 0$.

What is the range of torque for which conditions are linearly elastic? If there are no residual stresses, a torque $T = \tau_Y J/c = \tau_Y \pi c^3/2$ could be applied in either direction without yielding, for an elastic range of $\tau_Y \pi c^3$. If the residual stresses in Fig. 1.9-1d pre-

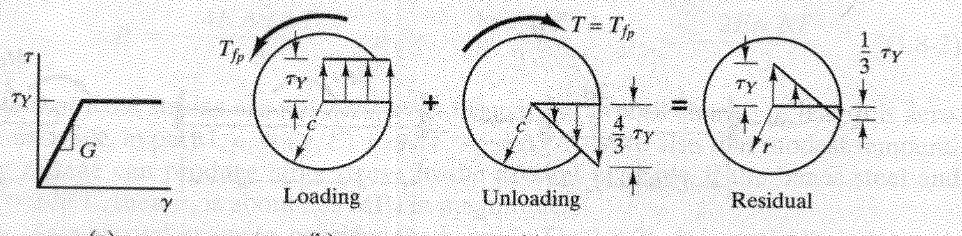

FIGURE 1.9-1 (a) Elastic–perfectly plastic material. (b,c,d) Stress distributions corresponding to fully plastic torque, unloading (reversed elastic) torque, and resultant (zero) torque.

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vail, we could apply a torque T_{fp} in the original direction or $(2/3)(\tau_Y\, J/c)$ in the reversed direction without renewed yielding, for an elastic range of $\tau_Y \pi c^3$. Thus the magnitude of the elastic range has not changed.

1.10 OTHER REMARKS

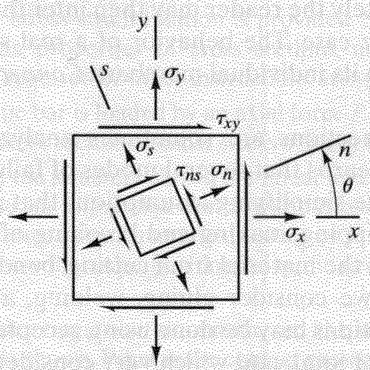
Stress Transformation. For reference, and for use in chapters that follow, two-dimensional stress transformation equations are shown in Fig. 1.10-1. These equations may be restated in other forms, for which the following trigonometric identities are useful.

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta \qquad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \tag{1.10-1}$$

Dimensional Homogeneity. In the calculation of stresses and deflections, it is often best to obtain a numerical result as the final step of solution, by substitution of data into a symbolic result. Thus we avoid manipulating numbers for some quantities that may cancel if manipulated as symbols. A more important reason is that a symbolic result permits a partial check on the correctness of the solution. A valid result is dimensionally homogeneous. For example, in Fig. 1.7-1b let $[v_L]$ and $[\theta_L]$ denote the respective dimensions of deflection and rotation. With F and L used here to denote dimensions of force and length respectively, dimensions of terms that contain M_L in the formulas of Fig. 1.7-1b are

$$[v_L] = \frac{(FL)L^2}{(F/L^2)L^4} = L \quad \text{and} \quad [\theta_L] = \frac{(FL)L}{(F/L^2)L^4} = 1 \tag{1.10-2}$$

These dimensions are correct: length units for v_L and dimensionless (radians in this case) for θ_L. This result does not prove the formulas to be correct, but had we obtained any other dimensions we would know for sure that the result is wrong.



$$\sigma_n = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\,\tau_{xy}\sin\theta\cos\theta$$

$$\sigma_s = \sigma_x\sin^2\theta + \sigma_y\cos^2\theta - 2\,\tau_{xy}\sin\theta\cos\theta$$

$$\tau_{ns} = \tfrac{1}{2}(\sigma_y - \sigma_x)\sin 2\theta + \tau_{xy}\cos 2\theta$$

FIGURE 1.10-1 Transformation of stresses in a plane.

Units. Example problems and homework problems serve as vehicles to convey concepts, principles, and procedures. Accordingly, the system of units used for numerical problems is of little importance. SI units are used in this book. Note that the average stress due to a 1 MN force on a square meter can be written in the following forms:

$$\sigma = \frac{10^6\text{N}}{1\ \text{m}^2} = 10^6\ \text{Pa} = 1\ \text{MPa} \qquad \text{or} \qquad \sigma = \frac{10^6\text{N}}{(1000\ \text{mm})^2} = 1\ \text{MPa} \qquad (1.10\text{-}3)$$

The latter form, which is used in subsequent chapters, avoids the conversion factor of 10^6. That is, forces in newtons, dimensions in millimeters, and stresses and moduli in megapascals form a consistent set of units, without need for conversion factors. However, if mass must be considered, as for inertia force loading, it will be easier to use meters rather than millimeters.

Classification by Problem Geometry. A slender member is usually called a *bar*, *beam*, or *shaft*, depending on whether the load is axial, lateral, or torsional. These problems are called one-dimensional, even though stress varies over a cross section as well as axially under bending or twisting load. A flat body whose thickness is much less than its other dimensions provides a two-dimensional problem. It is usually called a *plane* problem if loads have no lateral (thickness-direction) component, and a *plate* or *plate bending* problem if they do. In general, stresses in plane and plate problems vary with both of the in-plane directions. Stresses also vary in the thickness direction of a plate under lateral load. A floor slab is a familiar example. A *shell* is like a plate, but curved; familiar examples include an egg shell and a water tank. A shell can carry both surface-tangent and surface-normal loads. Many shells, and many solids too thick to be called shells, are symmetric about an axis and have loading that is also axisymmetric. Then nothing varies in the circumferential direction and analysis is simplified. Such a body is called a *shell of revolution* if it is thin-walled or a *solid of revolution* if it is not. An example of the latter is a turbine disk of strongly varying thickness that rotates at constant speed.

Connections. In this book, as in most other books about stress analysis, we may simply state that members are connected together, without saying how, and perhaps even disregarding stress concentrations associated with the connection. Thus we limit the scope of the book. Unfortunately the reader may then infer that connections are unimportant, which is far from the case. The behavior of a real structure may depend as much on its connections as on its individual members. *Connections are often the weakest parts of a stucture.*

Bolted or riveted connections are sometimes analyzed in a first course in mechanics of materials. One learns that several modes of failure are possible and that analysis can be tedious, despite simplifying assumptions that neglect stress concentrations, friction and possible slipping, making and breaking of contacts, misalignment, initial stresses, and damage to the material from cutting, bending, and punching holes. Other complexities arise if we consider gluing, welding, and shrink fits. Practical analysis and design of connections may be done using accepted codes and procedures that differ according to type of joint, and which vary considerably with type of indus-

From: Rapoff

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try. The study of connections is an important specialty in stress analysis. References include [1.1–1.5].

Handbooks. Many useful formulas for stress analysis do not appear in textbooks but may be found in handbooks or their computer software equivalents. The existence of this information does not erase the need for ability in stress analysis. Formulas can be used successfully only if the engineer understands the physical problem well enough to know what sort of formula to seek, understands the assumptions that underlie a formula, and is able to judge whether an answer produced by the formula is reasonable. Useful handbooks include [1.6, 1.7] for widespread coverage of stress and deflection, [1.8] for pressure vessels and the ASME code for them, [1.9] for buckling of bars, frames, plates, and shells, and [1.10] for modes and frequencies of vibration.

Codes. Engineering societies have produced codes that mandate allowable stresses, design procedure, and methods for testing, construction, operation, and maintainance of plants and equipment. Much of this information has grown out of experience with costly failures [1.11, 1.12]. Codes and specifications may receive little mention in engineering education, but it would be shortsighted to ignore them. Indeed, the engineer is often legally bound to follow one or more codes. Also, in situations where a code is applicable, it is likely to be the easiest route to an acceptable design. Students of structural engineering are probably familiar with design specifications of the American Institute of Steel Construction. There are a great many other codes and specifications, so many that space does not permit us to list them all.

The value of codes is illustrated by the history of boiler accidents. About the year 1900, on average, one boiler explosion occurred every day in the United States. Subsequently, codes for the design, construction, and operation of boilers were written and widely adopted. Today boiler explosions are rare despite a fifteen-fold increase in operating pressure since 1900 [1.13].

PROBLEMS

The following problems can be solved using the review material presented in this chapter, although many of the problems are less familiar or more challenging than those usually seen in an elementary textbook. Assume that materials are linearly elastic unless a nonlinear stress-strain relation is provided. State results symbolically in terms of loads, dimensions, properties of cross sections, and material constants, unless a numerical answer is required or other instructions are given.

1.4-1. A prismatic bar is loaded by an axial force P. Show that P must be directed through centroids of cross sections if axial stress σ is not to vary over a cross section.

1.4-2. Springs in the structure shown are linear and are unstressed when displacement v is zero. Determine an expression for v without assuming that $v \ll L$. With $L = 100$ mm and $k = 20$ N/mm, obtain numerical values of P for displacements v of 10 mm, 40 mm, and 50 mm. Show that superposition using the first two results does not yield the third. Plot P versus v.

1.4-3. Two slender rings, one aluminum and the other steel, just fit together at temperature $T = 0°C$, as shown. What is the contact pressure between them when $T > 0°C$?

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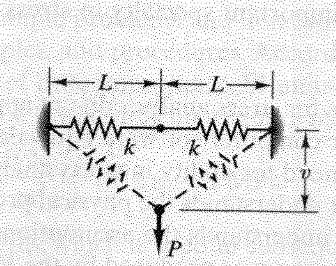
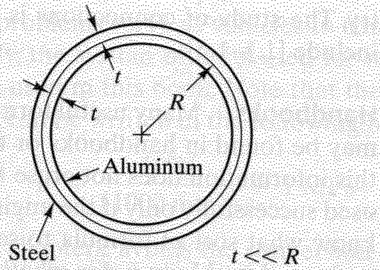
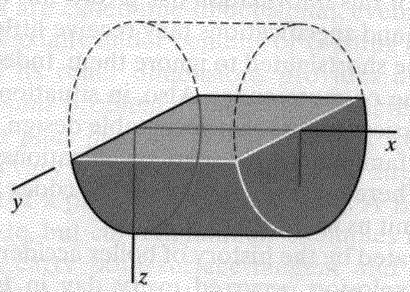


PROBLEM 1.4-2

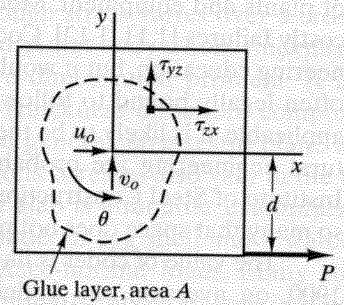


PROBLEM 1.4-3

1.5-1. A shaft of solid circular cross section is loaded by torque T. Consider a half-cylinder cut from the shaft by three cutting planes (see sketch). Show that stresses exposed by the cutting planes keep the half-cylinder in static equilibrium.



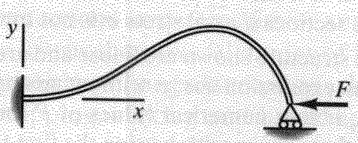
PROBLEM 1.5-1



PROBLEM 1.5-2

1.5-2. A flat plate is attached to a flat surface by a thin layer of glue of arbitrary shape and comparatively low modulus. An x-parallel load P is applied to the plate (see sketch). Axes xy are centroidal axes of the glue layer. What are shear stresses τ_{yz} and τ_{zx} in the glue layer? (Suggestion: Assume that these stresses are proportional to displacement components of the plate, and that the plate has rotation θ and translation components u_o and v_o at $x = y = 0$. Area A and its properties will appear in the solution.)

1.6-1. The sketch shows the post-buckling shape of a slender bar that was initially straight. Load F is known, and the shape $y = f(x)$ of the buckled bar is accurately known. What is the easiest way to determine support reactions at ends of the bar?



PROBLEM 1.6-1

1.6-2. In the beam of Fig. 1.6-1, it is proposed that flexural stress has the form $\sigma = ky$. Show that the cross-sectional area must have a zero product of inertia if this equation is to be correct.

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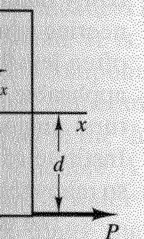

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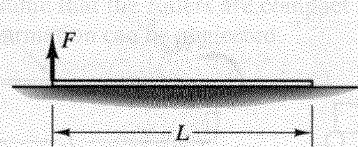
1.6-3. **(a)** Consider a prismatic beam, and conjecture that plane cross sections do *not* remain plane when bending moment is applied. Without equations, devise arguments that refute the conjecture.

(b) Similarly, consider a prismatic bar of circular cross section. Refute the conjectures that cross sections warp and radial lines become curved when torque is applied.

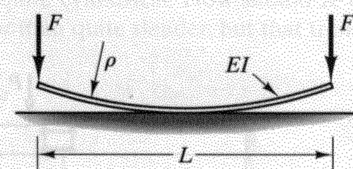
(c) The flexure formula $\sigma = My/I$ follows from the condition that plane cross sections remain plane in pure bending. A cantilever beam under transverse tip load experiences transverse shear deformation, and plane cross sections do *not* remain plane. Yet the flexure formula loses no accuracy. How can this be?

1.6-4. Let a prismatic beam have a rectangular cross section, b units wide and h units deep. The material has elastic moduli E_t in tension and E_c in compression. Derive expressions that relate stress to bending moment. The expressions should reduce to the conventional flexure formula if $E_t = E_c$.

1.6-5. The uniform beam shown has weight q per unit length. It rests on a rigid horizontal surface. If one end is lifted by a force $F < qL/2$, what is the maximum bending moment in the beam in terms of F and q?



PROBLEM 1.6-5



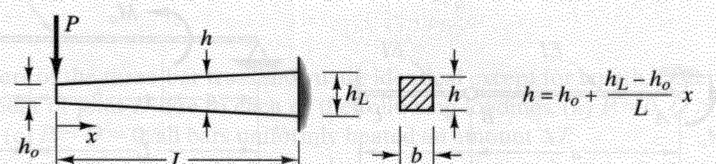
PROBLEM 1.6-6

1.6-6. When not loaded, the uniform beam shown has constant radius of curvature ρ, where $\rho \gg L$. Downward forces F are then applied to the ends.

(a) What value of F reduces curvature at the center of the beam to zero?

(b) For larger F, a central portion of length s becomes flat. Obtain an expression for s.

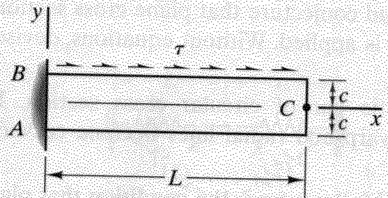
1.6-7. The beam shown has a slight taper. For what value of h_L/h_o does the largest flexural stress appear at $x = L/2$? What then is the ratio of flexural stress at $x = L/2$ to flexural stress at $x = L$?



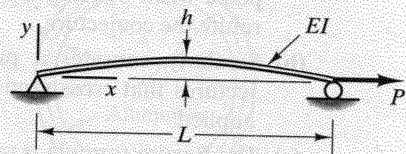
PROBLEM 1.6-7

1.7-1. A cantilever beam is loaded by uniform shear stress τ applied to its upper surface only, as shown. Obtain expressions for x-direction normal stress at A and at B. Neglect stress concentration effects. Also determine the deflection components of point C.

From: Rapoff



PROBLEM 1.7-1

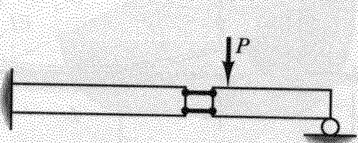


PROBLEM 1.7-2

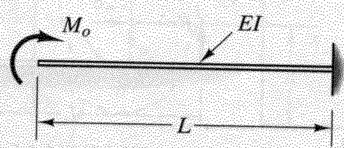
1.7-2. The slender bar shown is initially curved, so that with no load its axis has the equation $y = (4h/L^2)(Lx - x^2)$. What center deflection v_c is produced by force P? Assume that $v_c \ll h \ll L$.

1.7-3. Let the cantilever beam of Problem 1.7-1 be thermally loaded, such that the temperature varies linearly from ΔT on the lower surface to $-\Delta T$ on the upper surface. Obtain an expression for the deflection of point C due to ΔT.

1.7-4. It is proposed that a beam be constructed with a joint consisting of two horizontal links, as shown, so that the joint will transmit bending moment but no transverse shear force. Will this construction work as intended when load P is applied? Explain.



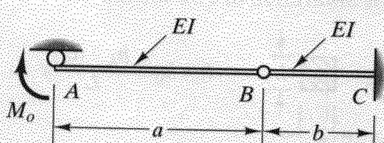
PROBLEM 1.7-4



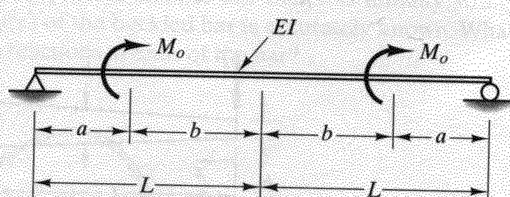
PROBLEM 1.7-5

1.7-5. The cantilever beam shown is so slender that its material remains linearly elastic even when displacements are large. Obtain expressions for the horizontal and vertical displacement components of the tip. Show that these expressions reduce to the expected small-deflection results when M_oL/EI is small.

1.7-6. For what value of a/b will the two parts of the beam shown have the same slope at hinge B when moment M_o is applied at A?



PROBLEM 1.7-6



PROBLEM 1.7-7

1.7-7. It is desired that both ends of a uniform beam remain horizontal when moments M_o are applied as shown. For what value of a/b will this be so?

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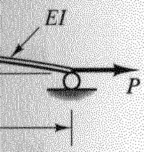

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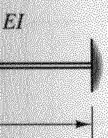

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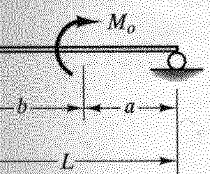

.7-7

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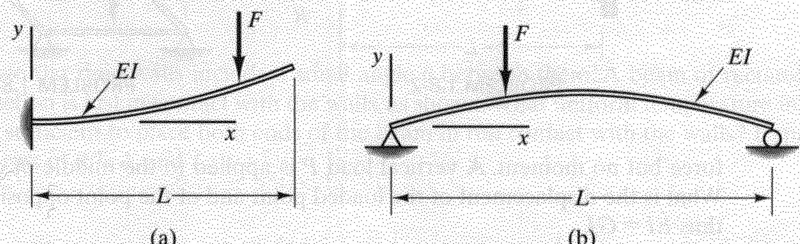
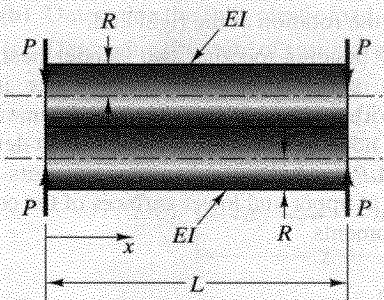
1.7-8. Each of the beams shown is to be made with a small initial curvature, such that a load F moving across the beam will have no vertical displacement. What should be the initial shape $y = f(x)$ of each beam?

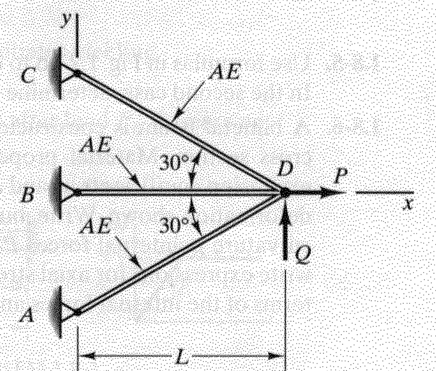


PROBLEM 1.7-8

1.7-9. Two identical rollers of average radius R are to be pushed together by end forces P, as shown. It is desired that the contact force between them be uniformly distributed along length L. Thus the rollers should not be quite cylindrical. How should R vary with x? Assume that the rollers are compact rather than quite slender, but that transverse shear deformation can be neglected.



PROBLEM 1.7-9



PROBLEM 1.8-1

1.8-1. Members of the three-bar truss shown are identical except for length. Determine the displacement of joint D due to each of the following loadings. (a) $P = 0$, $Q > 0$. (b) $P > 0$, $Q = 0$. (c) $P = Q = 0$; all bars uniformly heated an amount ΔT.

1.8-2. Let gears of different sizes be fastened to either end of a prismatic shaft of circular cross section. Let there be two such shafts, set parallel so that gears of radius R and $2R$ engage in the manner shown. Frictionless bearings, not shown, ensure that the shafts twist without bending. What torsional stiffness T/θ is seen by torque T?

1.8-3. A square frame is made by welding together four identical slender bars of circular cross section. The frame is placed horizontally atop corner supports that can exert vertical

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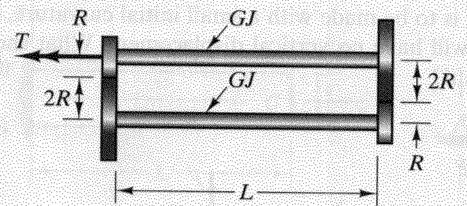
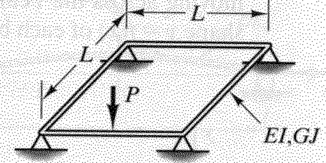


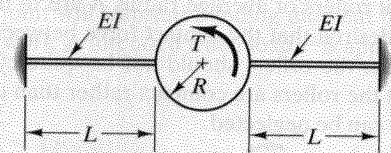
PROBLEM 1.8-2



PROBLEM 1.8-3

force but no moment. A vertical load P is applied to the middle of one side, as shown. What is the displacement of the loaded point and of the point opposite it? Let $G = E/2$; thus $EI = GJ$.

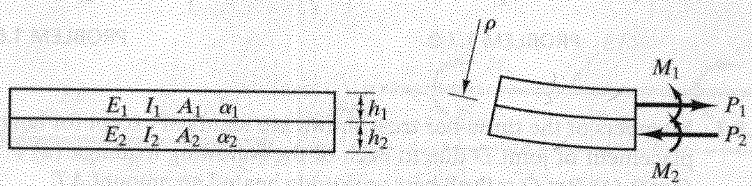
1.8-4. Two slender beams are built-in to a rigid disk and to rigid walls, as shown. Through what angle does the disk rotate if a small torque T is applied?



PROBLEM 1.8-4

1.8-5. Use formulas in Fig. 1.7-1b to determine the center deflection of each beam in Fig. 1.7-1c. In the second case, determine also the rotation at the right end.

1.8-6. A bimetal beam is constructed by bonding together two slender beams of rectangular cross section. Material properties of the component beams differ, including thermal expansion coefficients α_1 and α_2. With $\alpha_1 < \alpha_2$, uniform heating an amount ΔT causes the deformation shown. Write, but do not solve, sufficient equations to determine radius of curvature ρ, internal forces P_1 and P_2, and internal bending moments M_1 and M_2. Also, write expressions for axial stresses at upper and lower surfaces of the composite beam in terms of the internal forces and moments.



PROBLEM 1.8-6

1.8-7. Let several vertical posts of diameter D be arrayed in a straight line with distance L between them. A long slender beam is woven between the posts, as shown. Determine the maximum flexural stress in the beam.

1.8-8. A long straight beam has weight q per unit length. The beam is laid atop a small cylinder, as shown. Over what span $2L$ is the beam not in contact with the horizontal rigid floor?

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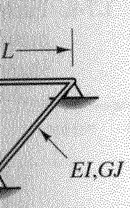

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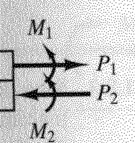

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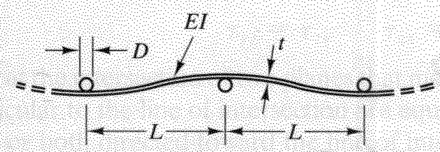
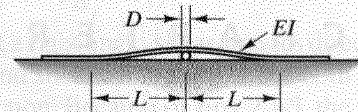
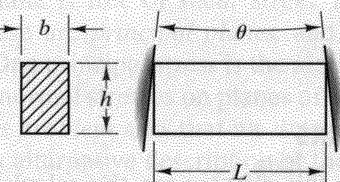


PROBLEM 1.8-7



PROBLEM 1.8-8

1.8-9. Two flat rigid walls include a small angle θ between them. A beam of rectangular cross section is just in contact with the walls, as shown. What uniform temperature increase ΔT is sufficient to place both ends of the beam in full contact with the walls? Express ΔT in terms of θ, h, α, and L.



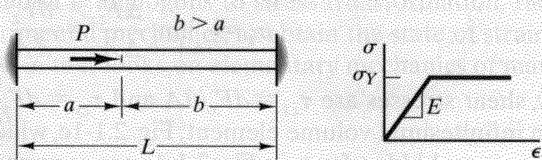
PROBLEM 1.8-9

1.9-1. The bar shown has uniform cross-sectional area A and is fixed at both ends. An idealized stress-strain relation is also shown. Assume that the relation is valid in compression as well as in tension.

(a) Determine the value of load P that initiates yielding.

(b) Determine the fully plastic load P_{fp}.

(c) Determine the state of residual stress after load P_{fp} is removed.



PROBLEM 1.9-1

1.9-2. Let the bar in Fig. 1.8-3a have the stress-strain relation used in Problem 1.9-1. Ends are fixed to the walls. Starting from the stress-free state, lower the temperature of the entire bar 1.5 times the amount ΔT that initiates plastic action.

(a) What then are the axial stresses? Express answers in terms of σ_Y.

(b) What are the residual stresses, and the residual displacement at the step, if the temperature is restored to its original value? Express answers in terms of σ_Y, L, and E.

1.9-3. For the three-bar truss of Problem 1.8-1, let $Q = 0$ and let the stress-strain relation be as depicted in Problem 1.9-1. Determine the fully plastic load P_{fp}. Also construct a dimensionless plot of P versus the horizontal displacement u_D of point D, using $P/A\sigma_Y$ as ordinate and $Eu_D/L\sigma_Y$ as abscissa.

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CHAPTER 1

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CHAPTER 2

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From: Rapoff